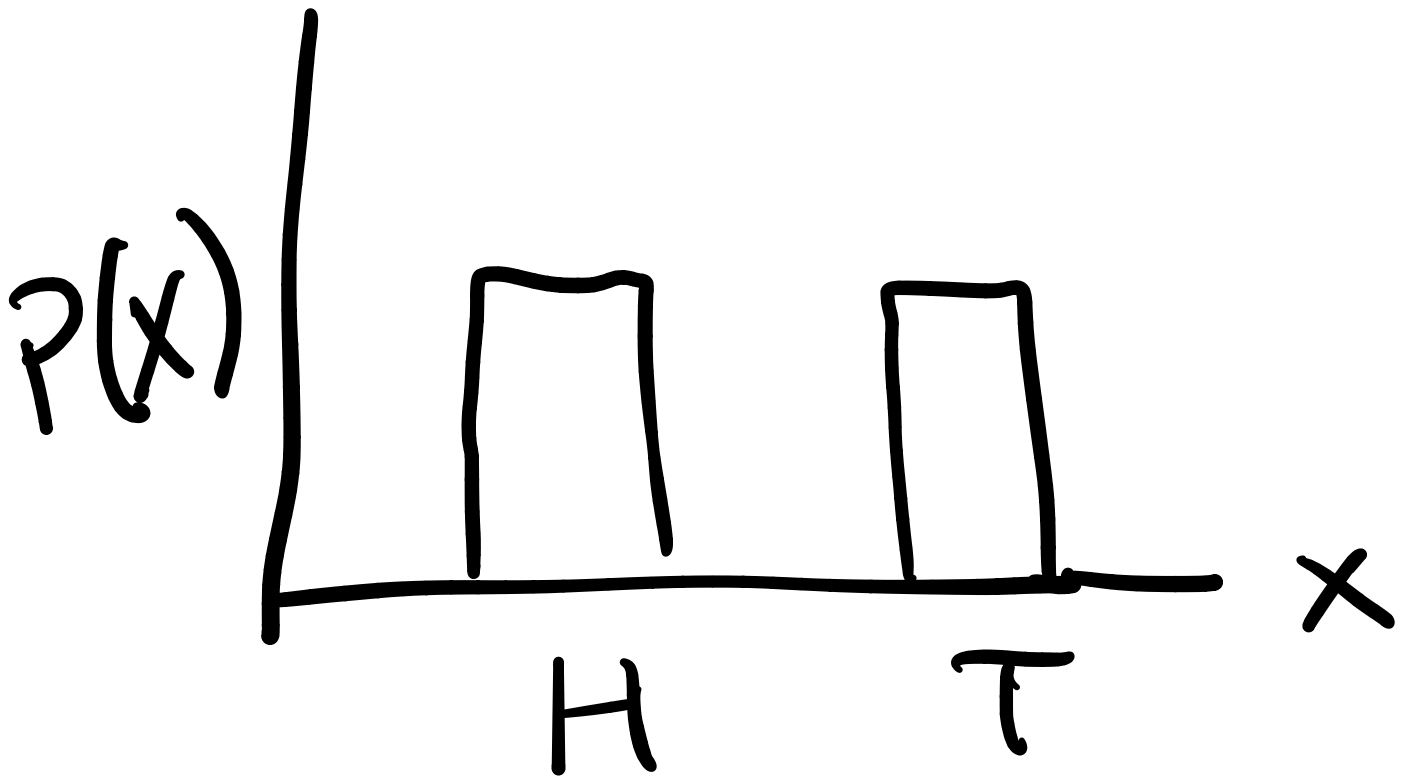
Lecture 2 – Probability

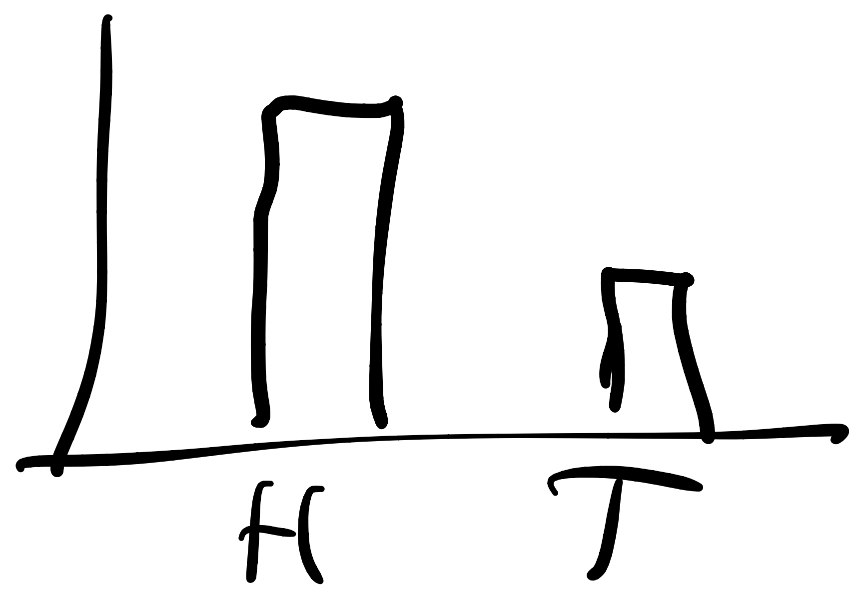
2 state systems: ↑↓   L/R

H & T   Flip coin:

2 outcomes, equally likely?!

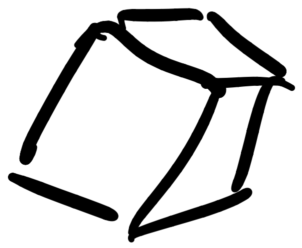


$\longleftrightarrow$



Exclusive events

H	T



1	2	3	4	5	6

Combine probabilities:

A and B for exclusive events = 0

A or B sum up probabilities

$$P(\text{odd}) = P(1 \text{ or } 3 \text{ or } 5) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6}$$

$$= 1/2$$

Multiple independent trials

flipping a coin many times

or flipping many coins

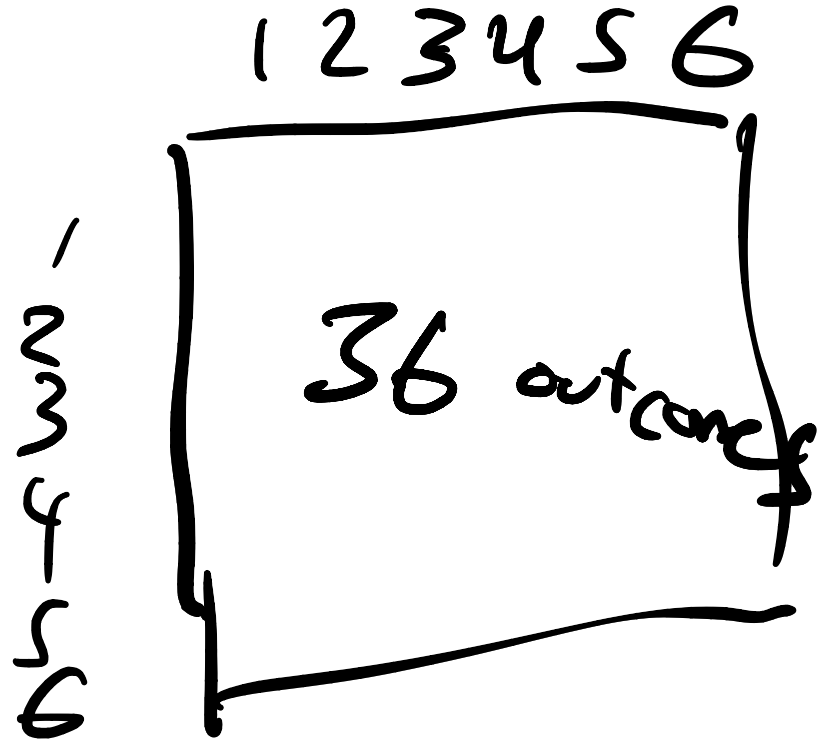
P (H and H and H)

probabilities multiply

$$= P(H)\,P(H)\,P(H) = \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} = 1/8$$

trial 1

trial 2	H	T
H	HH	TH
T	HT	TT



→ or addition

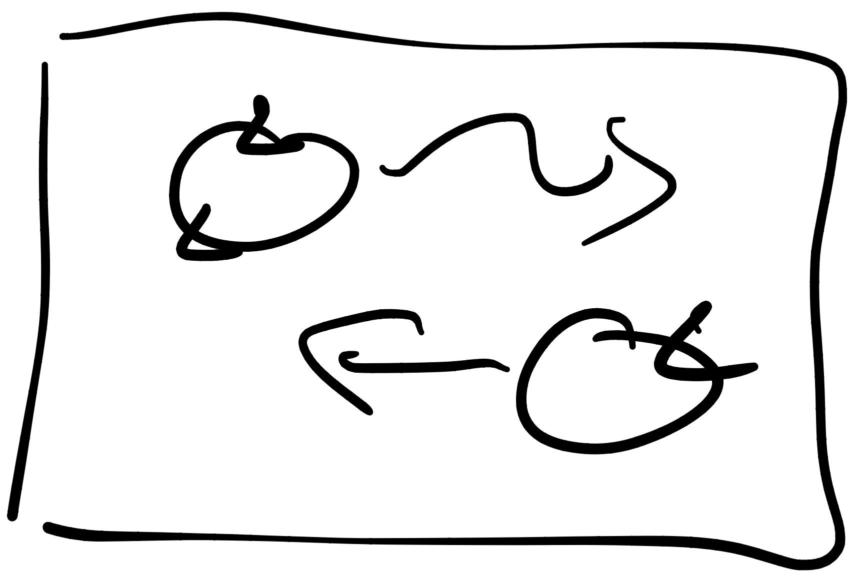
Are different "events" distinguishable

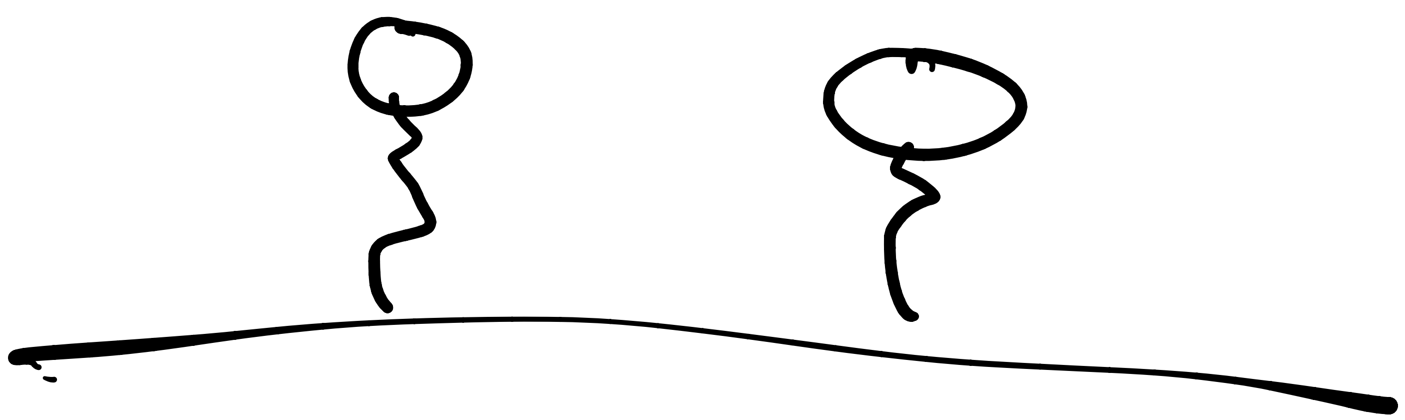
Example with 2 dice:

Chance of rolling 4 and 5

4 then 5 $\frac{1}{36}$

4 then 5 or 5 then 4 $\frac{2}{36}$



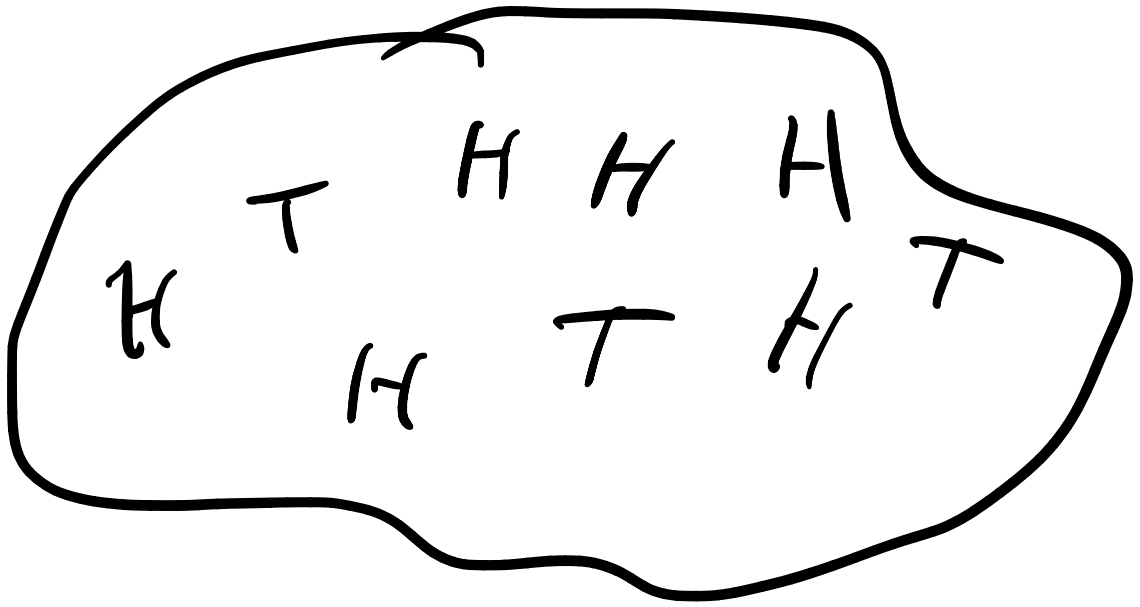


consider a long sequence / many molecules
2 states

$$\underbrace{H\,T\,H\,H\,T\,H\,T\,H\,H\cdots\cdots}_{}$$ Length N

Prob of this particular sequence

$$P(H)P(T)\,P(H)P(H)\cdots\cdots = \frac{1}{2}\cdot\frac{1}{2}\cdots\cdots = \frac{1}{2^N}$$

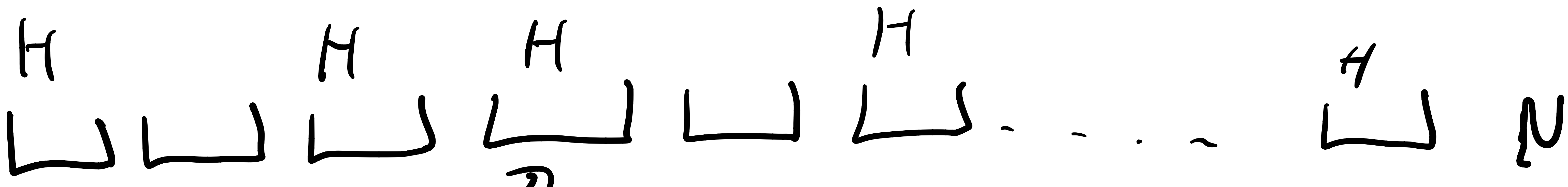


How many ways can I have M Hs in N molecules

Binomial coefficient $\binom{N}{m} = \dfrac{N!}{m!\,(N-m)!}$

N choose m

$$m! = m(m-1)(m-2)\cdots 1$$



m H's

N ways for first one

N-1 ways to put in second one

$$N\cdot(N-1)\cdot(N-2)\cdots(N-m) = \frac{N!}{(N-m)!} \leftarrow \text{distinguishable}$$

Indisting: $\dfrac{N!}{(N-m)!\,m!} = \binom{N}{m}$

Bernoulli triangle

$$\begin{array}{lc}
 & 1 \\
\genfrac{}{}{0pt}{}{1}{m} & 1 \quad 1 \\
\genfrac{}{}{0pt}{}{2}{m} & 1 \quad 2 \quad 1 \\
\genfrac{}{}{0pt}{}{3}{m} & 1 \quad 3 \quad 3 \quad 1 \\
\genfrac{}{}{0pt}{}{3}{m} & 1 \quad 4 \quad 6 \quad 4 \quad 1 \\
 & 1+5+10+10+5+1
\end{array}
\quad
\begin{array}{c} 0 \\ 0 \end{array}$$

$\leftarrow 32 = 2^5$

$$\sum_{m=0}^{N} \binom{N}{m} = 2^N$$

$\# = 5$

$$\begin{matrix} 0 & 0 & 0 & 0 & 0 \end{matrix}$$
$$\left.\begin{matrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{matrix}\right\} \binom{5}{1}$$

Binomial coefficient

$$(a+b)^N = \binom{N}{0} a^N b^0 + \binom{N}{1} a^{N-1} b^1 + \binom{N}{2} a^{N-2} b^2 \ldots$$

plug in $a=1$, $b=1$

$$2^N = \sum_{m=0}^{N} \binom{N}{m}$$

Preview multinomial

$$\frac{N!}{N_A!\, N_B!\, N_C! \cdots N_k!}$$

Probability of m out of N

$\Rightarrow$ binomial distribution

$P_H \leftarrow$ prob of heads $\quad P_T = 1 - P_H$

HTTTHH - ...

$$P(\text{indiv}) = P_H^{\,m}\, P_T^{\,N-m} = P_H^{\,m}(1-P_H)^{N-m}$$

mutiply by # sequences

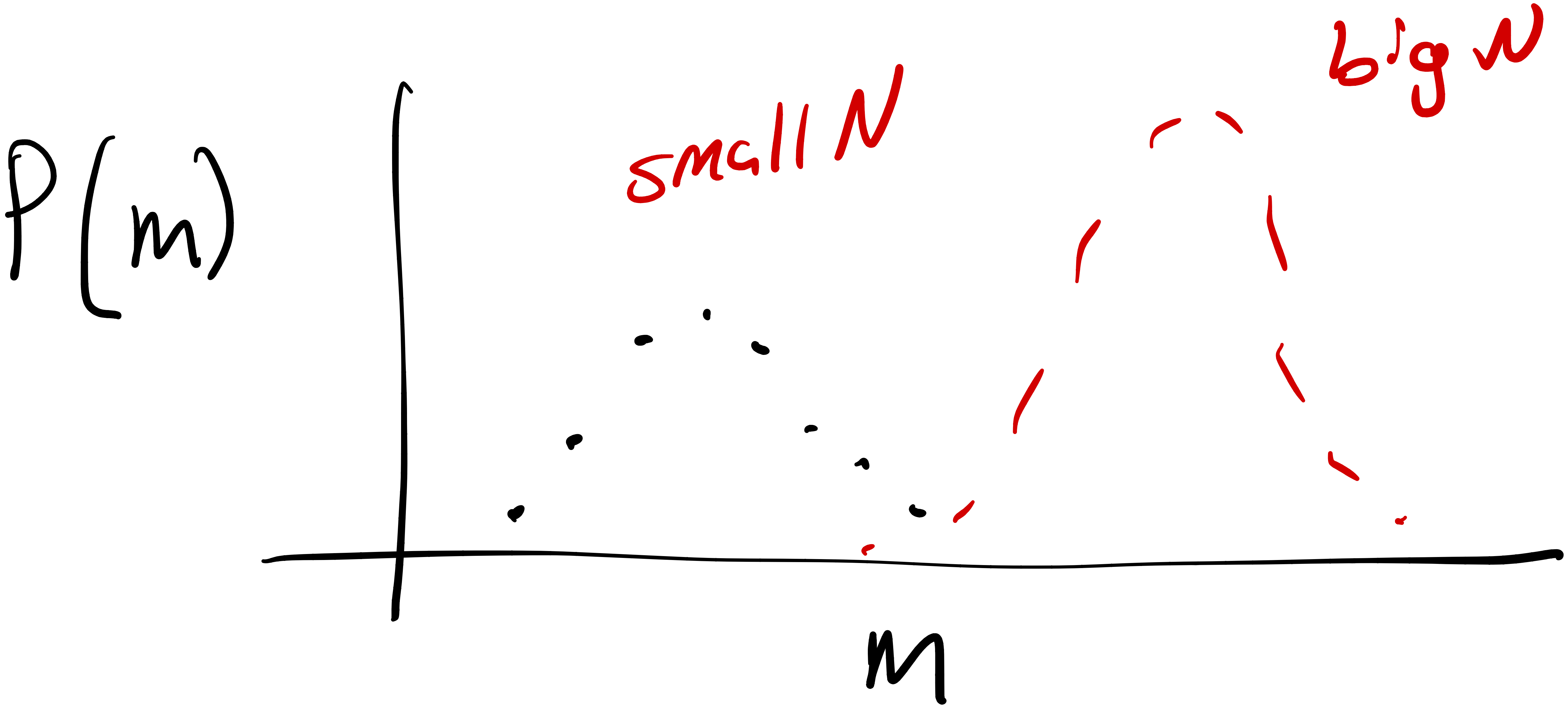
$$P(m;N) = \binom{N}{m} p^m (1-p)^{N-m}$$

binomial

probability distribution $\quad \sum_{m=0}^{N} P(m) = 1$

Binomial $1 = (p + (1-p))^N$ "normalized"

$$= \sum_{m=0}^{N} \binom{N}{m} p^m (1-p)^{N-m}$$



Characterize a distribution

what is average?
what is variance?

Binomial distribution

average, mean $\quad \mu_{distribution} = Np$

Variance $\doteq \sigma^2 = Np(1-p)$

$$\mu/\sigma \sim \frac{Np}{\sqrt{Np(1-p)}} \sim \frac{1}{\sqrt{N}}$$

formula for average

Sample average N observations

with outcomes $x_1, x_2, x_3 \ldots$

$$\mu = \frac{1}{N}\sum_{i=1}^{N} x_i$$

Mean of a distribution:

K is number of things that can happen

$$\mu = \sum_{i=1}^{K} k\,P(k)$$

eg

$$\sum_{i=1}^{K} k \binom{K}{i} p^{i}(1-p)^{K-i} = Kp$$

Variance
observations

first calc μ

$$\sigma^2 = \frac{1}{N}\sum_{i=1}^{N}(x_i - \mu)^2$$

avg squared deviation

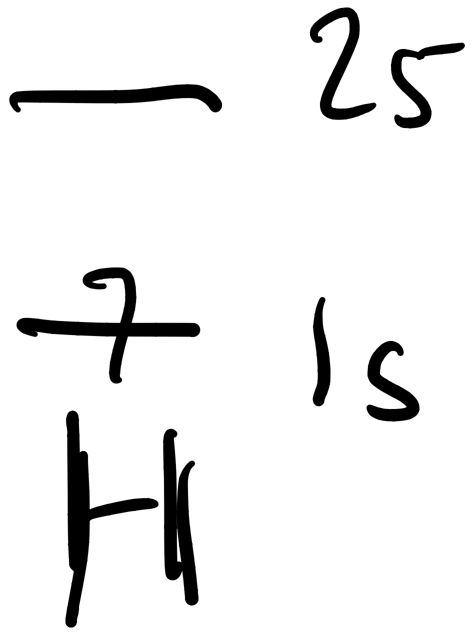
distribution: $$\sigma^2 = \sum_{i=1}^{K}(x_i - \mu)^2 P(x_i)$$

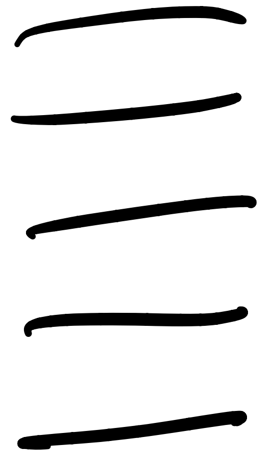
binomial dist $kp(1-p)$

HW

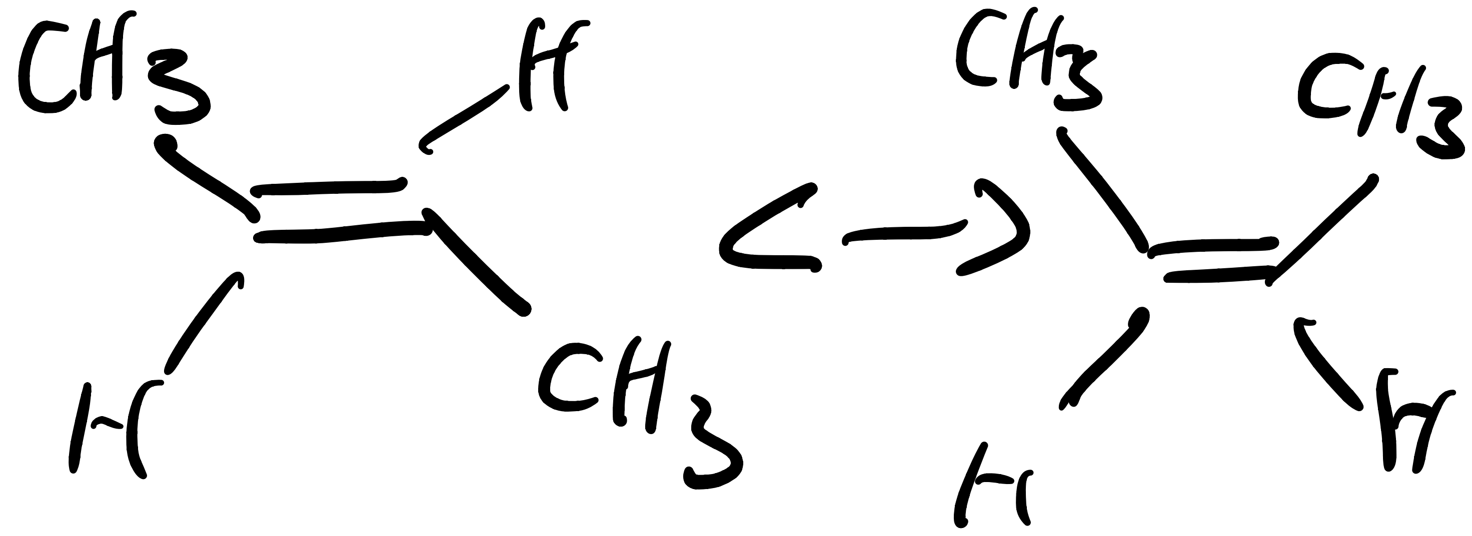
Equiv: $$\sigma^2 = \langle X^2\rangle - \langle x\rangle^2 \geq 0$$

μ

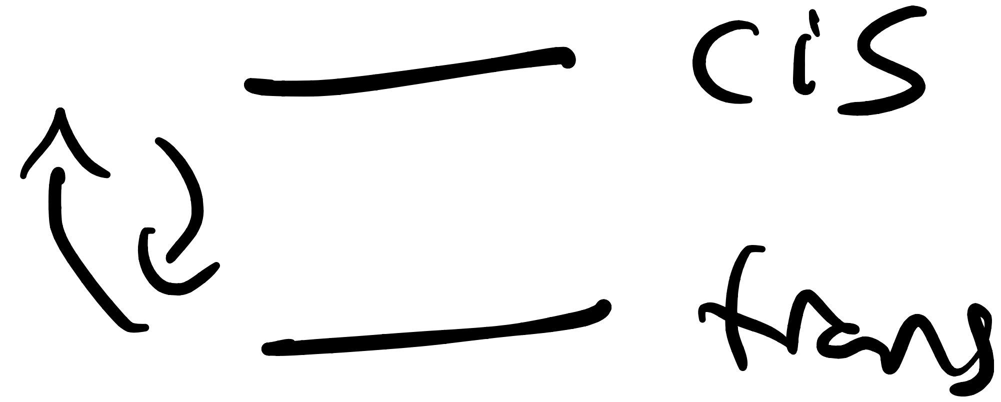


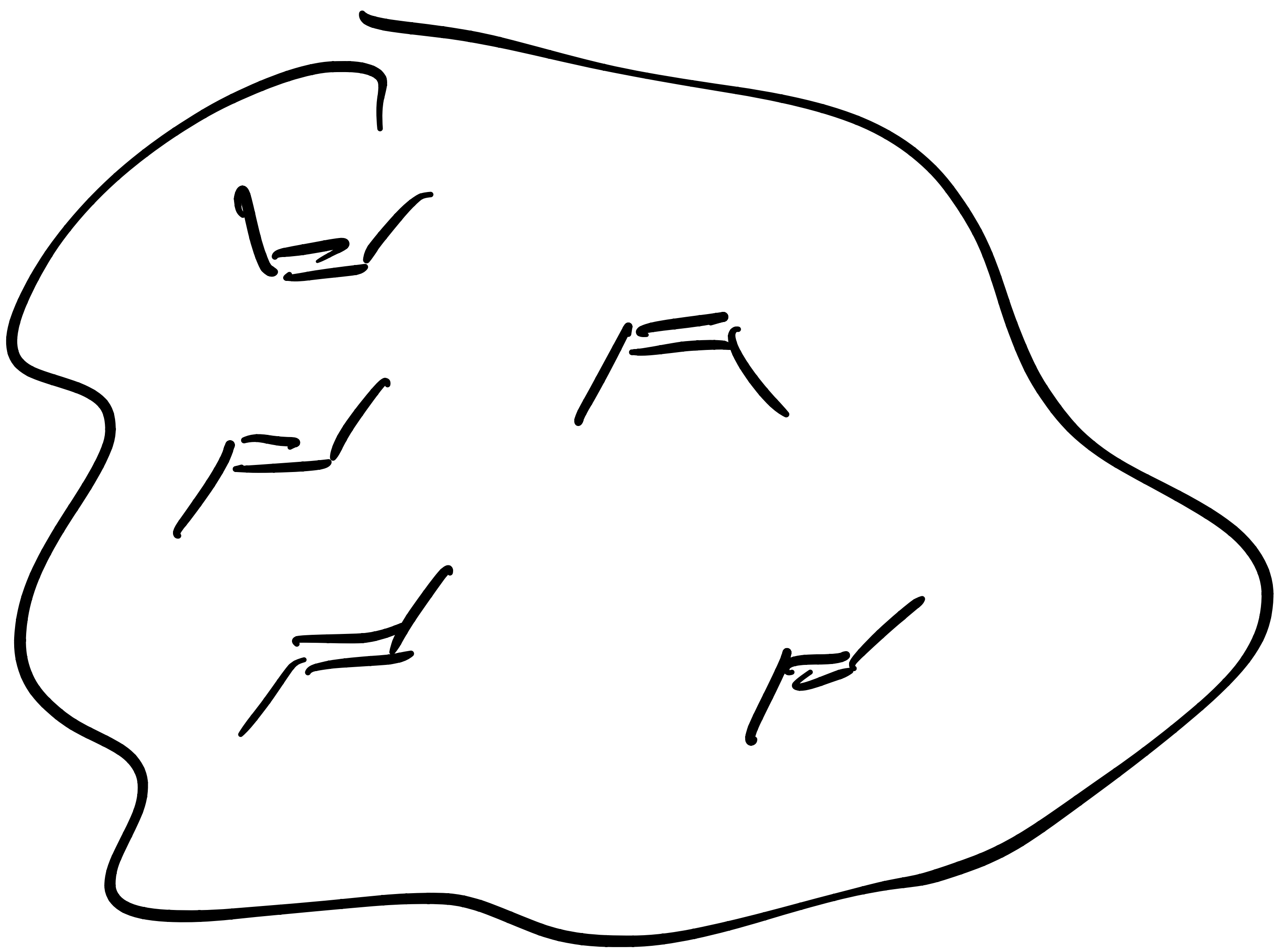


Cl-Cl



lower E　　　higher E





$$P(i) = \frac{e^{-\varepsilon_i / k_B T}}{Z}$$

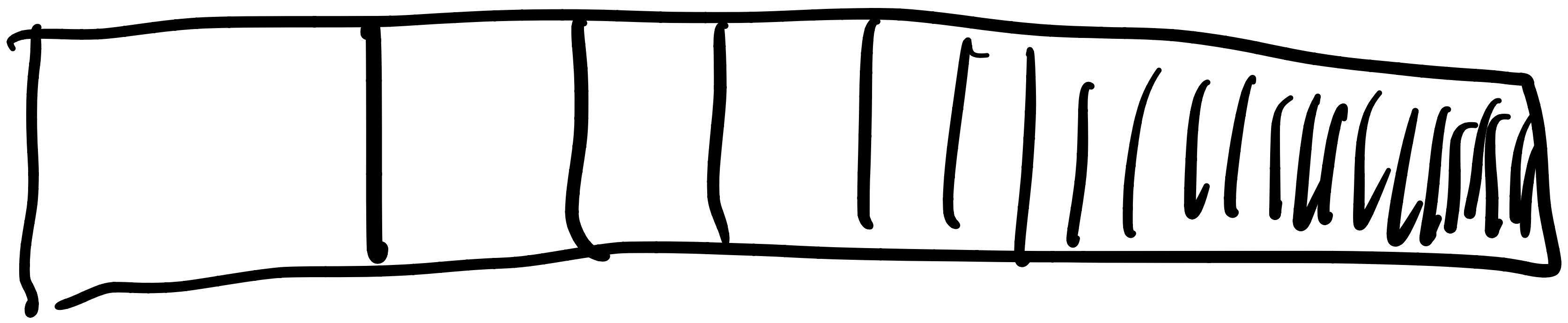
Z "partition function"

$Z \in$ partition function

$$\sum_{i=1}^{k} e^{-\varepsilon_i/k_B T} / Z = 1$$

$$Z = \sum_{i=1}^{k} e^{-\varepsilon_i/k_B T}$$

Boltzmann factors



Z

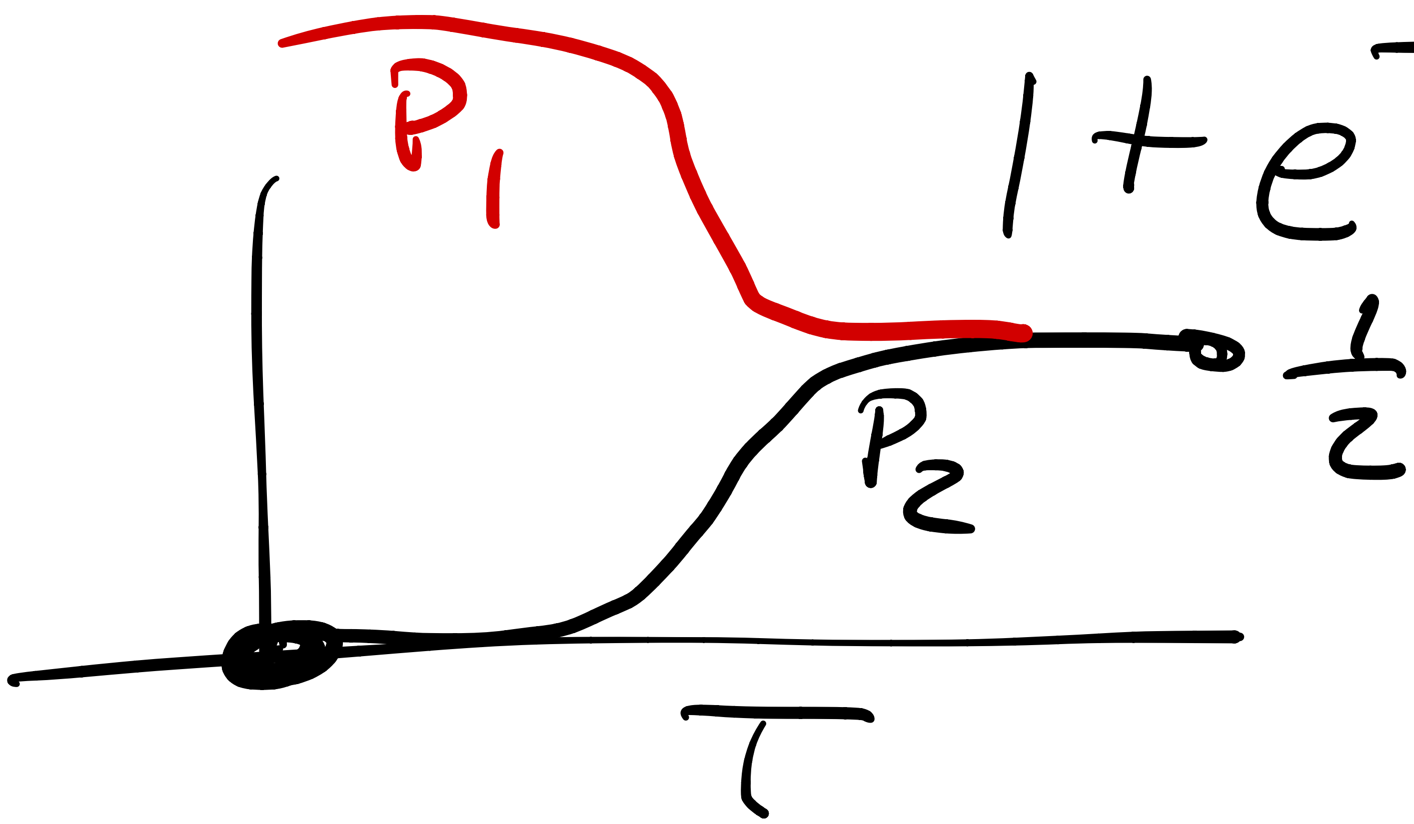
for 2 states:

$$P(i) = \frac{e^{-\varepsilon_i/k_B T}}{e^{-\varepsilon_0/k_B T} + e^{-\varepsilon_i/k_B T}}$$

$$P(i) = \frac{e^{-\varepsilon_i/k_B T}}{e^{-\varepsilon_1/k_B T} + e^{-\varepsilon_2/k_B T}}$$

$$P(\text{state } 2) = \frac{e^{-\varepsilon_2/k_B T}}{e^{-\varepsilon_1/k_B T} + e^{-\varepsilon_2/k_B T}}$$

$$= \frac{1}{1 + e^{-(\varepsilon_2 - \varepsilon_1)/k_B T}}$$



$$P(N,N) = \binom{N}{N} P_2^{\,N}$$

$$= P_2^{\,N}$$

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