Calculus - Quick Reminder & connection to prob

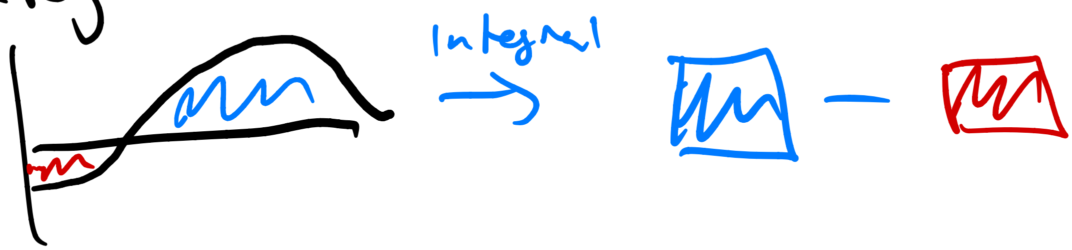
This time, a few new concepts before thermo

1. derivative is slope of function at a point
2. max/min/saddles occur where

$$df/dx = 0$$

can check d^2f/dx^2 or draw to determine which

3. Integrals are "area under curve"



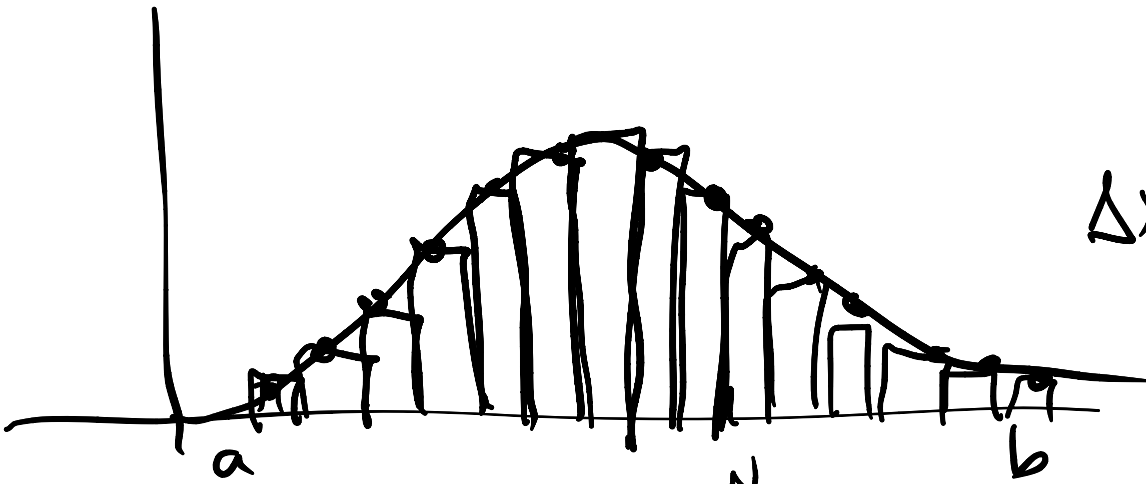
4. Integrals are anti-derivatives

$$\int_a^b \frac{df}{dx} = f(b) - f(a)$$

Derivatives lose info, so

$\int \frac{df}{dx}\,dx = f(x) + C$, family of functions

⑤ Integrals of cts functions like sum over discrete function



$$\Delta x = \frac{b-a}{N_{bins}}$$

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{N} f(x_i)\,\Delta x = \frac{b-a}{N} \sum_{i=1}^{N} f(x_i)$$

Note: Δx has units, and so does "dx" so integrals multiply on a unit and derivative divides one out

Cts prob distributions:

eg $P(t) = \frac{1}{\tau} e^{-t/\tau}$ $t \in (0, \infty)$, $\mu = \tau$, $\sigma^2 = \tau^2$

$P(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}$ $\mu = \mu$, $\sigma^2 = \sigma^2$

$$\langle x^n \rangle = \int_{-\infty}^{\infty} x^n P(x)\, dx$$

$$\& \int_{-\infty}^{\infty} P(x)\, dx = 1$$

$$\mathrm{Var}(x) = \langle x^2 \rangle - \langle x \rangle^2$$

$$= \int x^2 P(x)\, dx - \left(\int x P(x) \right)^2$$

$$= \int (x - \langle x \rangle)^2 P(x)\, dx$$

$$\mathrm{Prob}(x \le a) = \int_{-\infty}^{a} P(x)\, dx$$

$$\mathrm{Prob}(x \ge a) = \int_{a}^{\infty} P(x)\, dx$$

Other special rules

Chain rule for derivatives

$$\frac{d}{dx} f(g(x)) = g'(x)\, f'(g(x))$$

example:

$$\left.\begin{array}{l} f(x) = x^2+2 \\ g(x) = (x-1) \end{array}\right\} \begin{array}{l} f' = 2x \\ g' = 1 \end{array}$$

$$\frac{d\, f(g(x))}{dx} = 1 \cdot 2(x-1) \quad \text{by chain rule}$$

explicitly $f(g(x)) = (x-1)^2 + 2$

$$= x^2 - 2x + 3$$

$$\frac{d}{dx} f(g(x)) = 2x - 2 = 2(x-1) \checkmark$$

Ex2: $\frac{d}{dx} \frac{1}{g(x)} = \frac{d}{dx}[f(g(x))]$ $\quad f(x) = 1/x \quad f' = -\frac{1}{x^2}$

$$= -g'(x) \cdot \frac{1}{g(x)^2}$$

product rule

$$\frac{d}{dx}(f(x)g(x)) = f(x)g'(x) + f'(x)g(x)$$

New topic: differentials

derivative of $f(x)$ is written $\frac{df}{dx}$

This is because it comes from the limit $\lim_{b \to a} \frac{f(b)-f(a)}{b-a} = \lim_{\Delta x \to 0} \frac{f(a+\Delta x)-f(a)}{\Delta x}$

$\Delta x = b-a$

We learn not to treat $\frac{df}{dx}$ as an actual fraction, ie if $\frac{df}{dx} = m$,

don't write $df = m\,dx$

But actually, if we are careful, we can do this, and it is called "differentials"

It is explicitly saying if we make a super tiny change to x, how does f respond. The size of the response is df/dx (sensitivity coeff)

[consider spring $F = kx$]

Note df & dx are not zero, (unless $\frac{df}{dx} = 0$ then f is)

Useful notation for thermo!

Chain rule infinitesimals

$$d(fg) = g\,df + f\,dg$$

Integration by parts:

Integrate above (careful about limits)

$$\int d(fg) = \int g\,df + \int f\,dg$$

$$\Rightarrow \int f\,dg = fg - \int g\,df$$

aka $\int u\,dv = uv - \int v\,du$

Partial Derivatives - multiple variables

Partial derivatives means we take a derivative w.r.t. one variable, and hold others constant

Eg $f(x,y) = (x^2+1)y^2$

$$\left(\frac{\partial f}{\partial x}\right)_y = y^2 \cdot \frac{\partial (x^2+1)}{\partial x} = y^2 \cdot 2x$$

$$\left(\frac{\partial f}{\partial y}\right)_x = (x^2+1)\frac{\partial y^2}{\partial y} = (x^2+1)\cdot 2y$$

Global maxs & mins come when all partial derivs = 0 simult.

For regular functions "cross derivs" agree

$$\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x} f(x,y)\right) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y} f(x,y)\right) = \frac{\partial^2}{\partial x \partial y} f(x,y)$$

Eg - prev example - both equal $4xy$ ✓

Differentials of multiple Variables

Partial derivatives give sensitivity to changing single variable at a time

$$df(x,y) = \left[\frac{\partial}{\partial x} f(x,y)\right]_y dx + \left[\frac{\partial}{\partial y} f(x,y)\right]_x dy$$

and so forth for more vars

This relates to Taylor series

$$f(x+\Delta x) = f(x) + \Delta x f'(x) + \frac{\Delta x^2}{2!} f''(x) + \dots$$

$$= \sum_{n=0}^{\infty} \frac{\Delta x^n}{n!} f^{(n)}(x)$$

$$df \approx f(x+\Delta x) - f(x) \approx f'(x)\Delta x$$ for small Δx

For multivariables, beginning of taylor series is

$$f(x+\Delta x, y+\Delta y) \simeq f(x,y) + \Delta x \frac{\partial f}{\partial x}(x,y) + \Delta y \frac{\partial f}{\partial y}(x,y)$$

$$\underbrace{+ \frac{\Delta x^2}{2}\frac{\partial^2 f}{\partial x^2}(x,y) + \frac{\Delta y^2}{2}\frac{\partial^2 f}{\partial y^2}(x,y) + \Delta x \Delta y \frac{\partial^2 f}{\partial x \partial y}(x,y) + \dots}_{\text{small}}$$

2 ways

To thermo: integration adds lots of small changes together, along a path in x & y

$$\int_{x_1 y_1}^{x_2 y_2} df = f(x_2, y_2) - f(x_1, y_1)$$

$$= \int_{\substack{\text{path } x_1 \to x_2 \\ y_1 \to y_2}} \left(\frac{\partial f}{\partial x}\right) dx + \left(\frac{\partial f}{\partial y}\right) dy$$

for functions that can be written as exact differential

$\Rightarrow$ called a state function, only need to know endpoints

Ex: Energy, Entropy, not: heat, work!