Lecture 2 - Probability & Distributions

Reminder: Last time independent outcome, same event

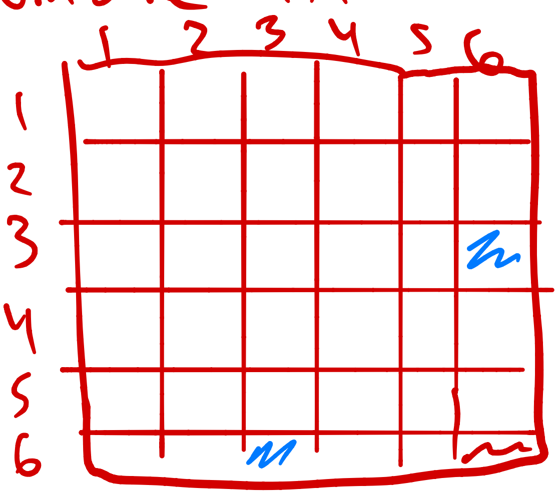
$P_{A \cap B} = 0 \qquad P_{A \cup B} = P_A + P_B$

Independent events:

Example: 2 people each roll a die

what is prob of 2 sixes?

Combine into a single event to see rule



$1/36$ possible events

general rule for ind outcomes

$P_{A \cap B} = P_A \times P_B$

Be careful depending on question. what about prob of $\{5, 6\}$

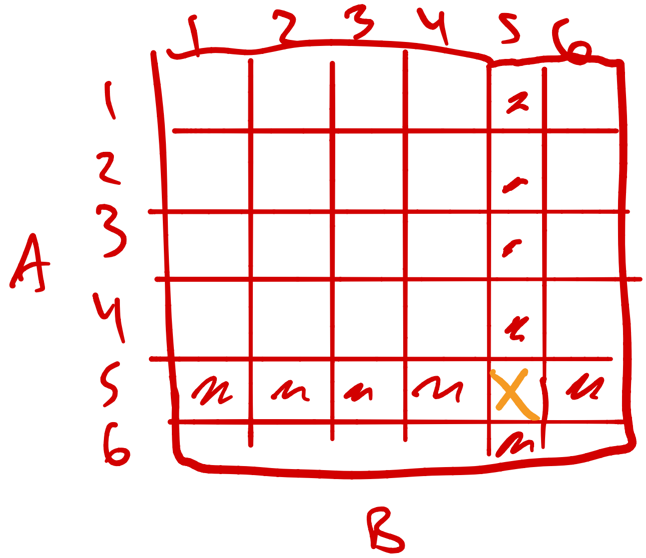
$P_{\text{person1-6} \cap \text{person2-5}} = 1/36$

but also opposite case $= 1/36$

so prob of "scoring" $= 2/36 = 1/18$

(this is an example of the "or" rule for the joint outcomes, add area)

What about prob that one rolls 5 or the other rolls 5 (either, not both)



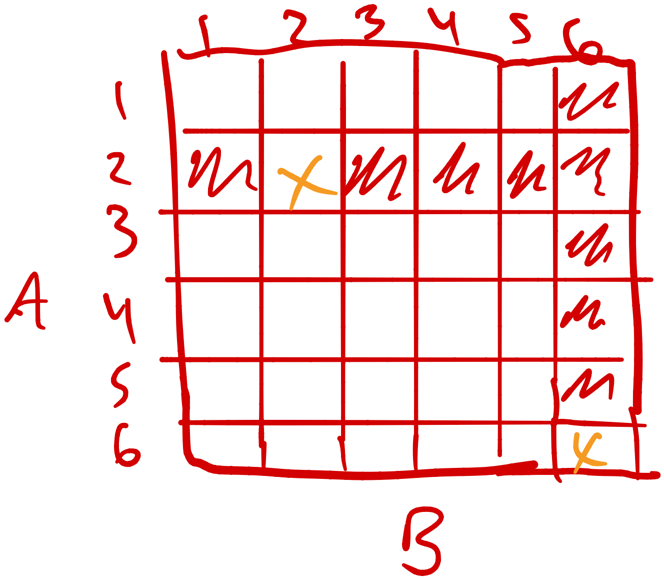
$$P_{A=5 \cup B=5} = P_{A=5} + P_{B=5} - P_{A=5 \cap B=5}$$

$$= 1/6 + 1/6 - \frac{1}{36}$$

$$= 11/36$$

What about $A=2$ or $B=6$

but not $A=B=2$
$A=B=6$



$$P_{A=2 \cap B=6}$$

$$= P_{A=2} + P_{B=6} - P_{A=2} \cap P_{B=2} - P_{A=6} \cap P_{B=6}$$

Not "~" prob event X doesn't happen $= 1 - P_X$ ✱

For a sequence of observations $\{o_i\} = \{x_1, x_2, x_3 \ldots x_N\}$

$P_{o_1 = x_1 \cap o_2 = x_2 \cap \ldots \cap o_N = x_N} = \prod_{i=1}^{N} P_{x_i}$

notation

[worksheet, 5 min]

How does order matter?

→ How many ways to rearrange N objects → Turns out:

$$N! = N(N-1)(N-2) \cdots 1$$

This is because, imagine N slots

First item has N choices, second item N-1, and so forth, until full

Returning to coin flips, a sequence would be like $H, T, T, H, T, \dots$

prob would be $P_{o_1=H \cap o_2=T, \dots} = P_H \cdot P_T P_T P_H P_T \cdots$

Every seq is unique $= P_H^{N_H} P_T^{N_T}$

$= P_H^{N_H} (1-P_H)^{N-N_H}$

mutual exclusive

What if we just want to know how many seqs of length N have N_H heads

Then these could come in any order, and the prob of N_H/N is much higher

What is # ways to order N_H & N_T items?

ordering $\boxed{N_H} + \boxed{N_T}$ coins in N slots

W.L.O.G. Put in N_H coins, as before

$N \cdot (N-1) \cdot (N-2) \cdots$ but only down to $N - N_H$

$= \dfrac{N!}{N_H!}$

In every case there are N_T slots filled w/ T's, which can go in $N_T!$ order

Each of these sequences is identical if indistinguishable so

$$N \text{ choose } N_T = \frac{N!}{N_H!\,N_T!} = \frac{N!}{N_H!\,(N-N_H)!} = \frac{N!}{(N_T!)(N-N_T)!}$$

Symmetric!

These values, written $\binom{N}{M}$ or N choose M are called "binomial" coefficients b/c they are the terms in expansion

$$(a+b)^N = a^N + \binom{N}{1} a^N b^{N-1} + \dots = \sum_{i=0}^{N} \binom{N}{i} a^i b^{N-i}$$

$$\mathrm{Prob}(N_A; N) = \binom{N}{N_A} P_A^{N_A} (1-P_A)^{N-N_A}$$

↑ binomial distribution

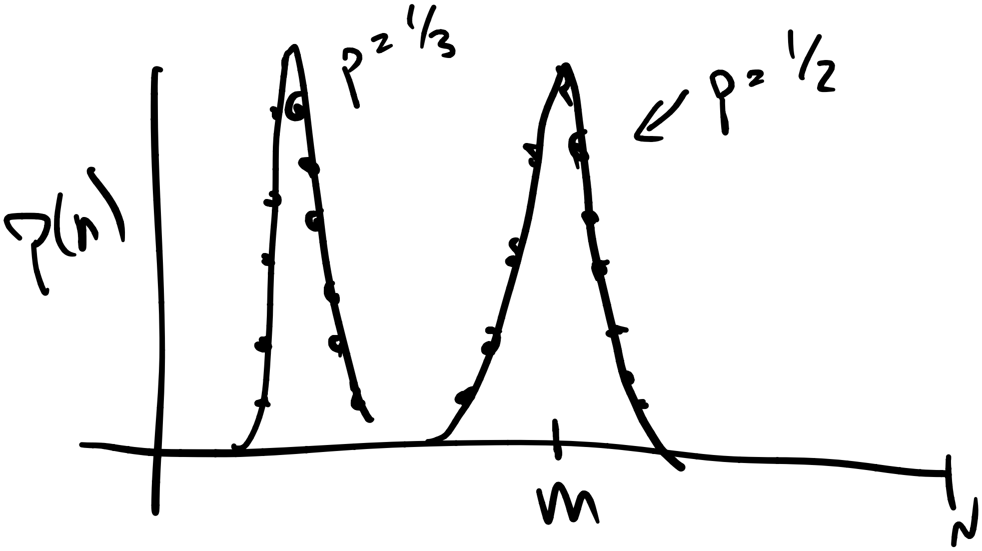
Normalized: $1 \stackrel{?}{=} \sum_{N_A=1}^{N} \binom{N}{N_A} P_A^{N_A} P_B^{N_B} = (P_A+P_B)^N = 1^N = 1$ ✓

Familiar terms:

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 1 & & 2 & & 1 & \\ 1 & & 3 & & 3 & & 1 \end{array}$$

← pascal's triangle

Key: Meaning is, probability of exactly m successes in N trials (Binom(N,m))



★ mean and variance

Mean is simple average

Know, if $\{X_1, X_2, X_3 \ldots\}$, average is

$\bar{\mu} = \frac{1}{N}\sum_{i=1}^{N} x_i$ ← this is a sample mean, it is computed from data

If we have a distribution of "X's"

$\mu = \sum_{n=1}^{N} X_i \, P(X_i)$ also written $\langle X \rangle$

Another important quantity is Variance, σ^2

b/c σ = std dev

$\sigma^2 = \langle (X - \langle x \rangle)^2 \rangle$

$= \langle x^2 \rangle - \langle x \rangle^2 = \sum_{n=1}^{N} (X_i - \mu)^2 P(X_i)$

★

For data $\sigma^2 = \frac{1}{N}\sum_{n=1}^{N} (x_i - \bar{\mu})^2$ explain

assumed X_i is "sampled" from dist

Binomial dist

$\mu = Np$ $\sigma^2 = Np(1-p)$

$\Rightarrow \sigma/N \sim 1/\sqrt{N}$ so dist gets more rel narrow

Poisson Distribution

return to cts distr

Key: prob of a number of random events happens in fixed interval (usually time)

Like number of decay events of radioactive nuclei per hour or number of proteins in some area in a membrane

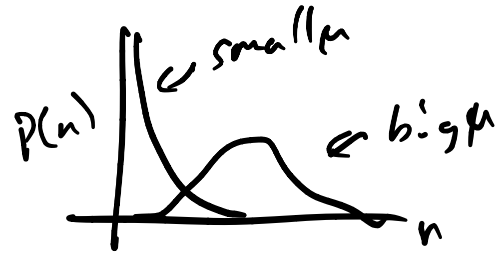
Comes from Binom $N \to \infty$ trials
"rare events" $p \to 0$

$$p(n, \mu) = \frac{\mu^n e^{-\mu}}{n!}$$ where μ is avg number expected

Eg $\rho_{protein} = 1/\mu m^2$ look at $100\ nm^2$ area

$$\mu = \rho A = \frac{100\ nm^2}{1\times10^6\ nm^2} = 1\times10^{-4}$$

most likely 0!



$$\boxed{\sigma^2 = \mu}$$

$\sigma/\mu \sim 1/\sqrt{\mu}$ also