# Lecture 18 - Phase transitions, pt 3

$$\cdots \uparrow \downarrow \uparrow \downarrow \uparrow \uparrow \downarrow \uparrow \downarrow \uparrow \downarrow \cdots$$

$$\mathcal{H} = -J \sum_{i=1}^{N} s_i s_{i+1} - h \sum_{i=1}^{N} s_i$$

$$= \sum_{i=1}^{N} \Big( \underbrace{-J s_i s_{i+1} - \frac{h}{2}(s_i + s_{i+1})}_{\mathcal{H}(s_i, s_{i+1})} \Big)$$

$$Z = \sum_{\{s_i\}} e^{-\beta \sum_{i=1}^{N} \tilde{\mathcal{H}}(s_i, s_{i+1})} \qquad e^{-\beta \mathcal{H}(s_i, s_{i+1})}$$

$$P_{s,s'} = e^{\beta J s s' + \beta \frac{h}{2}(s + s')}$$

$$Z = \sum_{s_1} \sum_{s_2} \sum_{s_3} \cdots \sum_{s_N} e^{\beta[J s_1 s_2 + h(s_1+s_2)/2]} \times e^{\beta[J s_2 s_3 + h(s_2+s_3)/2]} \cdots$$

$$= \sum_{s_1} \sum_{s_2} \cdots \sum_{s_N} P_{s_1,s_2} \cdot P_{s_2,s_3} \cdot P_{s_3,s_4} \cdots P_{s_N,s_1}$$

$$\langle s_1 | P | s_2 \rangle \langle s_2 | P | s_3 \rangle \cdots \langle s_N | P | s_1 \rangle$$

$$P = \begin{bmatrix} \overset{\uparrow\uparrow}{e^{\beta J + \beta h}} & \overset{\uparrow\downarrow}{e^{-\beta J}} \\ \underset{\downarrow\uparrow}{e^{-\beta J}} & \underset{\downarrow\downarrow}{e^{\beta J - \beta h}} \end{bmatrix}$$

$$Z = \mathrm{Tr}\left[ P^N \right]$$

$$= \mathrm{Tr}\left[ D^N \right]$$

$$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

$$= \lambda_1^N + \lambda_2^N$$

eigenvalues of P?

$$\begin{pmatrix} e^{\beta(J+h)} & e^{-\beta J} \\ e^{-\beta J} & e^{\beta(J-h)} \end{pmatrix}$$

$$0 = \mathrm{Det}\left[ P - \lambda I \right]$$

$$= \begin{vmatrix} e^{\beta h + \beta J} - \lambda & e^{-\beta J} \\ e^{-\beta J} & e^{\beta J - \beta h} - \lambda \end{vmatrix}$$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb$$

$$0 = \left( e^{\beta J + \beta h} - \lambda \right)\left( e^{\beta J - \beta h} - \lambda \right) - e^{-2\beta J}$$

$$\lambda_{\pm} = e^{\beta J}\cosh(\beta h) \pm e^{\beta J} \times \sqrt{\sinh^2(\beta h) - e^{-4\beta J}}$$

$$Z = \lambda_+^N + \lambda_-^N$$

$$\left[\lambda_1^N + \lambda_2^N + \lambda_3^N \cdots \lambda_n^N\right]$$

$Z \to \lambda_+^N$ as $N$ gets large

$$\cosh(\beta h) = \frac{1}{2}\left(e^{\beta h} + e^{-\beta h}\right)$$

$$Z = \lambda_1^N + \lambda_2^N + \lambda_3^N \cdots \lambda_n^N$$

$$= \lambda_1^N \left( 1 + \left(\frac{\lambda_2}{\lambda_1}\right)^N + \left(\frac{\lambda_3}{\lambda_1}\right)^N + \cdots + \left(\frac{\lambda_n}{\lambda_1}\right)^N \right)$$

$$\lim_{N \to \infty} a^N \quad \text{if} \quad |a| < 1 \quad \text{is } 0$$

$$F(N, \beta, h) \approx -k_B T \ln\left(\lambda_+^N\right)$$

$$= -\frac{N}{\beta} \ln(\lambda_+)$$

$$m = \frac{k_B T}{N} \frac{\partial \ln Z}{\partial h} \approx k_B T \frac{\partial \ln \lambda_+}{\partial h}$$

$$= \frac{\sinh(\beta h) + \left(\sinh^2(\beta h) - e^{-2\beta J}\right)^{-1/2} \cdot \sinh(\beta h)\cosh(\beta h)}{\cosh(\beta h) + \sqrt{\sinh^2(\beta h) - e^{-2\beta J}}}$$

as $h \to 0$?

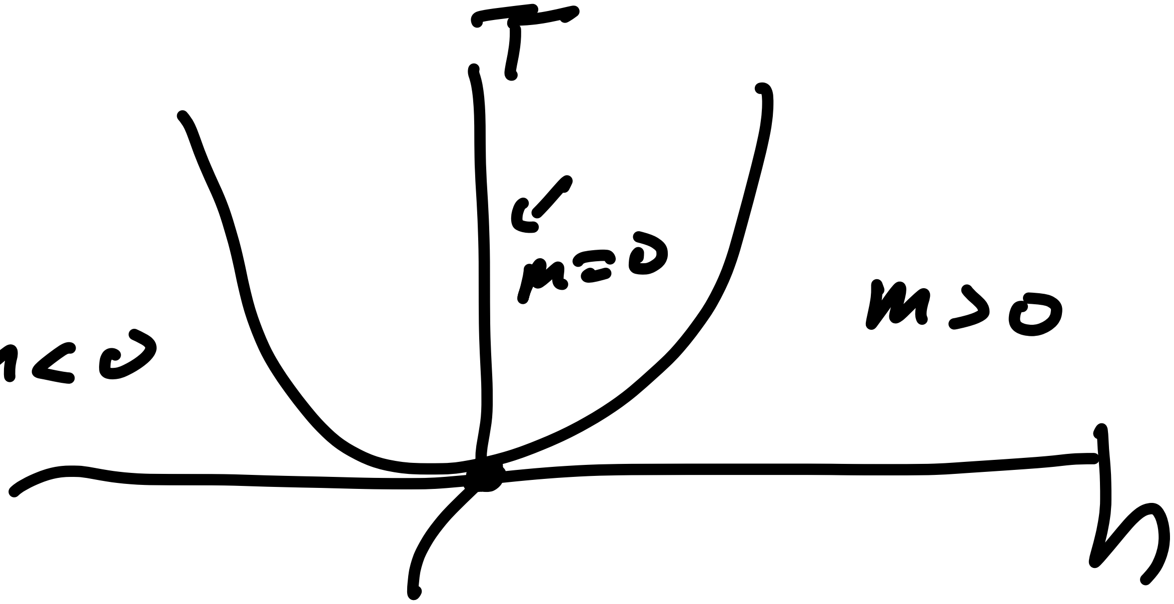
$\cosh(\beta h) \to 1$

$\sinh(\beta h) \to 0$

$\lim_{h\to 0} m(h) = 0$ if $\beta < \infty$

$\beta \to \infty$ $\quad m \Longrightarrow \frac{\sinh(\beta h) \pm \cosh \beta h}{\cosh(\beta h) \pm \sinh(\beta h)} = \pm 1$

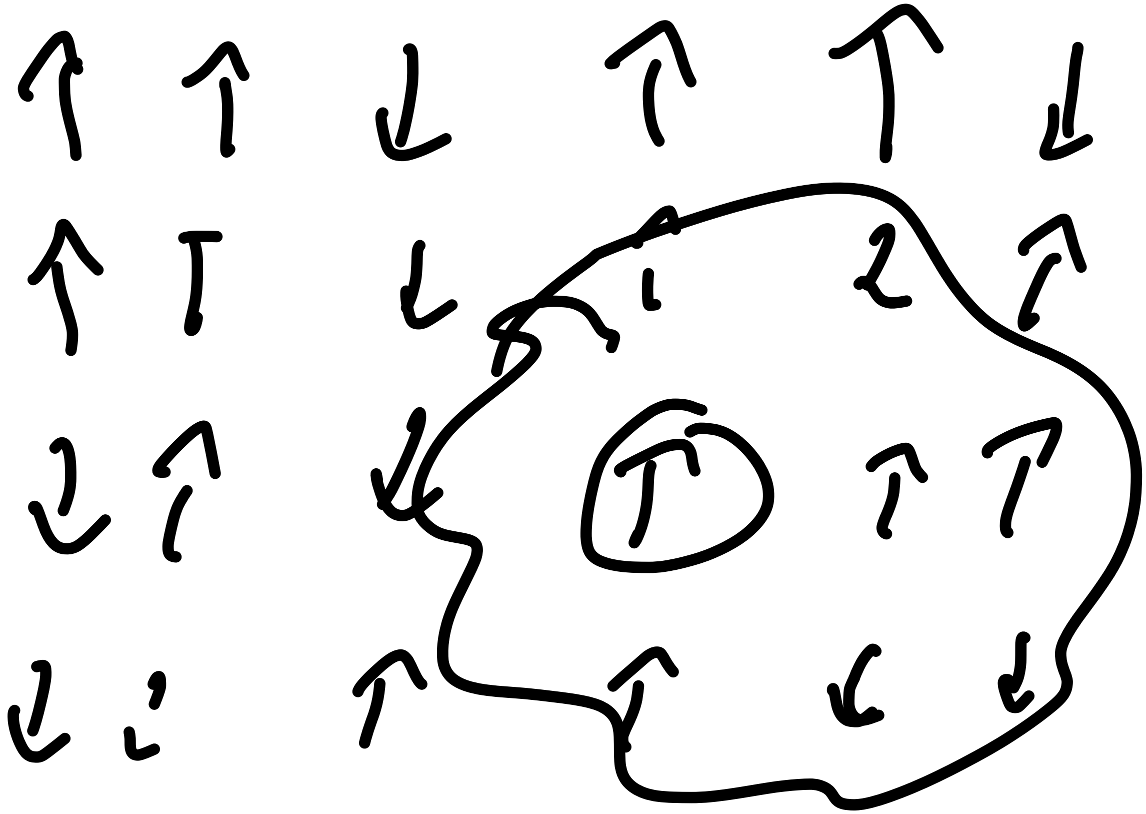
1d




$T=0, m=\pm 1$

"infinite" dimensions

mean field



$m = \langle \sum s_i \rangle /$

$\#$neigh $= z$

avg spin $\to m$

$$\mathcal{H}_{nd} = \sum_{\langle ij \rangle} -J s_i s_i - h \sum_i s_i$$

$$\approx -2 z J \sum_i (s_i \cdot m) - \sum_i h s_i$$

$$H_{1d} = -\frac{J}{2}\sum_{\langle ij\rangle} s_i s_j - h\sum_{i=1}^{N} s_i$$

$$\delta s_i = s_i - m \qquad \langle \delta s_i \rangle = 0$$

$$s_i = m + \delta s_i$$

$$= -\frac{J}{2}\sum_{\langle ij\rangle}(m+\delta s_i)(m+\delta s_j) - h\sum s_i$$

$$= -J/2 \sum_{\langle ij\rangle}\left(m^2 + m(\delta s_i + \delta s_j) + \cancel{\delta s_i \delta s_j}\right) - h\sum_{i=1}^{N} s_i$$

$$= -J/2 \sum_{\langle ij \rangle} \left(m^2 + m(\delta s_i + \delta s_j) + \cancel{\delta s_i \delta s_j}\right) - h \sum_{i=1}^{N} s_i$$

$$\approx -J/2 \sum_{\langle ij \rangle} \left(m^2 + m(s_i - m) + m(s_j - m)\right) - h \sum_{i=1}^{N} s_i$$

$$\left(-m^2 + m s_i + m s_j\right) - h \sum_{i=1}^{N} s_i$$

$$= +Jzm^2N - \frac{Jm}{2} \underbrace{\sum_{\langle ij \rangle} (s_i + s_j)} - h \sum_{i=1}^{N} s_i$$

$$= +Jzm^2N - 2mJz \sum_{i=1}^{N} s_i - h \sum_{i=1}^{N} s_i$$

$$\sum_{\langle i,j \rangle} = \sum_{i=1}^{N} \sum_{j=1}^{N} \underbrace{\text{neigh}(i,j)}_{\substack{1 \text{ if } i,j \\ 0 \text{ otherwise}}} \quad x$$

$$= 2 \sum_{i=1}^{N} (z \cdot x)$$

$i = 1 \;,\; j = N, 1$ $\qquad$ $i = 2 \;,\; j = 1, 3$

$j = 1 \;,\; i = N, 1$ $\qquad$ $j = 2 \;,\; i = 1, 3$

$$\mathcal{H}_{MF} = Jm^2Nz - (h + 2mJz)\sum_{i=1}^{N} S_i$$

$$Z = \sum_{S_1, S_2, \ldots, S_N} e^{-\beta\mathcal{H}_{MF}(S_1, S_2 \ldots S_N)}$$

$$= e^{-\beta Jm^2Nz}\left[\sum_{S_i} e^{+\beta(h+2mJz)S_i}\right]^N$$

$$= e^{-\beta Jm^2Nz}\left[2\cosh(\beta(h+2mJz))\right]^N$$

$$m = k_BT/N \; \partial \ln Z/\partial h$$

$$Z_{MF} = e^{-\beta J m^2 N z/2} \left[ 2\cosh(\beta(h+2mJz)) \right]^N$$

$$m = \frac{k_B T}{N} \frac{\partial \ln Z_{MF}}{\partial h}$$

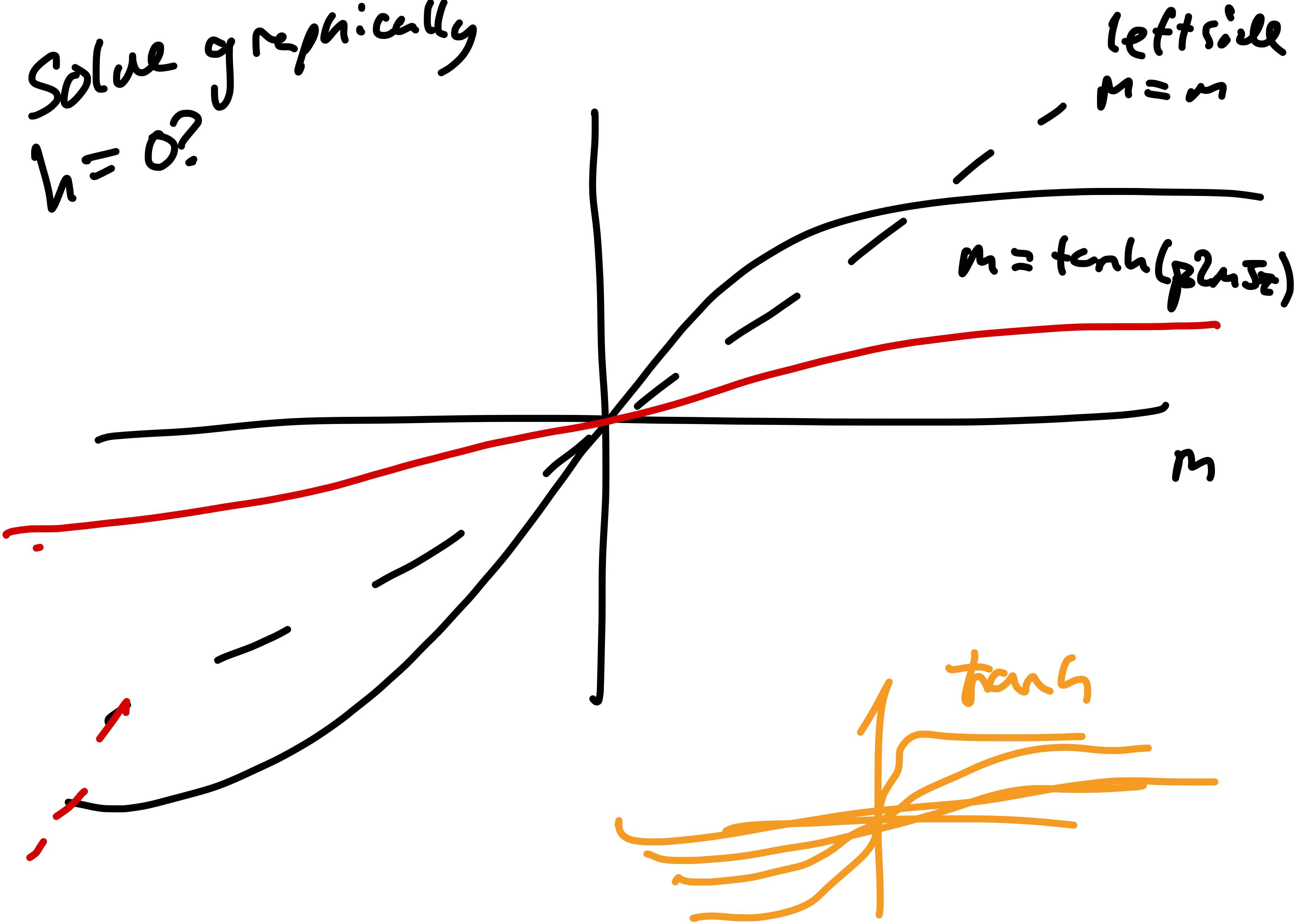
$$m = k_B T \frac{\partial}{\partial h} \ln\left[2\cosh(\beta(h+2mJz))\right]$$

$$= k_B T \cdot \frac{2\sinh(\beta(h+2mJz))}{2\cosh(\beta(h+2mJz))} \cdot \beta$$

$$m = \tanh\left[\beta(h+2mJz)\right] \quad \star$$

Solve graphically

$h = 0$?



2 solutions w/ $m > 0$

at $2\beta J z > 1$

$$k_B T < 2 J z$$

$$T_c^{mf} = 2Jz/k_B \qquad z = 2d$$

$$= 4Jd/k_B$$

neglecting fluctuations → overestimate of $T_c$