Phase transitions pt 2

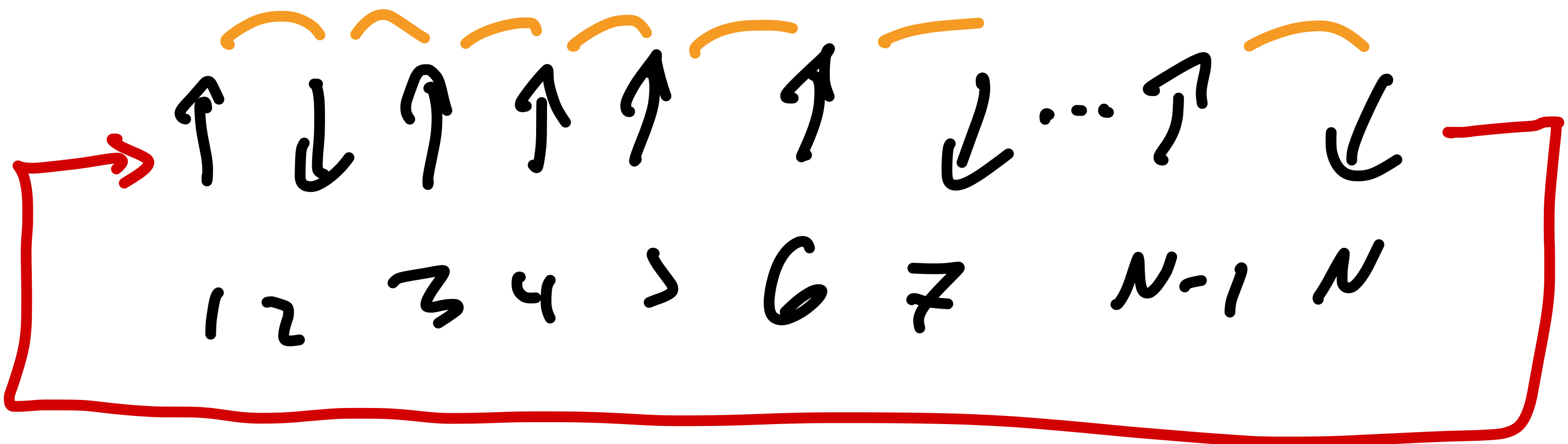
$$\mathcal{H} = -J \sum_{\langle ij \rangle} s_i s_j - h \sum_{i=1}^{N} s_i$$

$$s_i = \begin{cases} +1 \\ -1 \end{cases}$$

$$\uparrow \; \downarrow \; \uparrow \; \downarrow \; \uparrow \; \downarrow \; \uparrow$$

$$\sum_{\langle ij \rangle} = \sum_{i=1}^{N} \sum_{j \in (i-1,\, i+1)} -J s_i s_j = 2 \sum_{i=1}^{N} -J s_i s_{i+1}$$

$$\mathcal{H}_{1d} = -J\sum_{i=1}^{N} s_i s_{i+1} - h\sum_{i=1}^{N} s_i$$

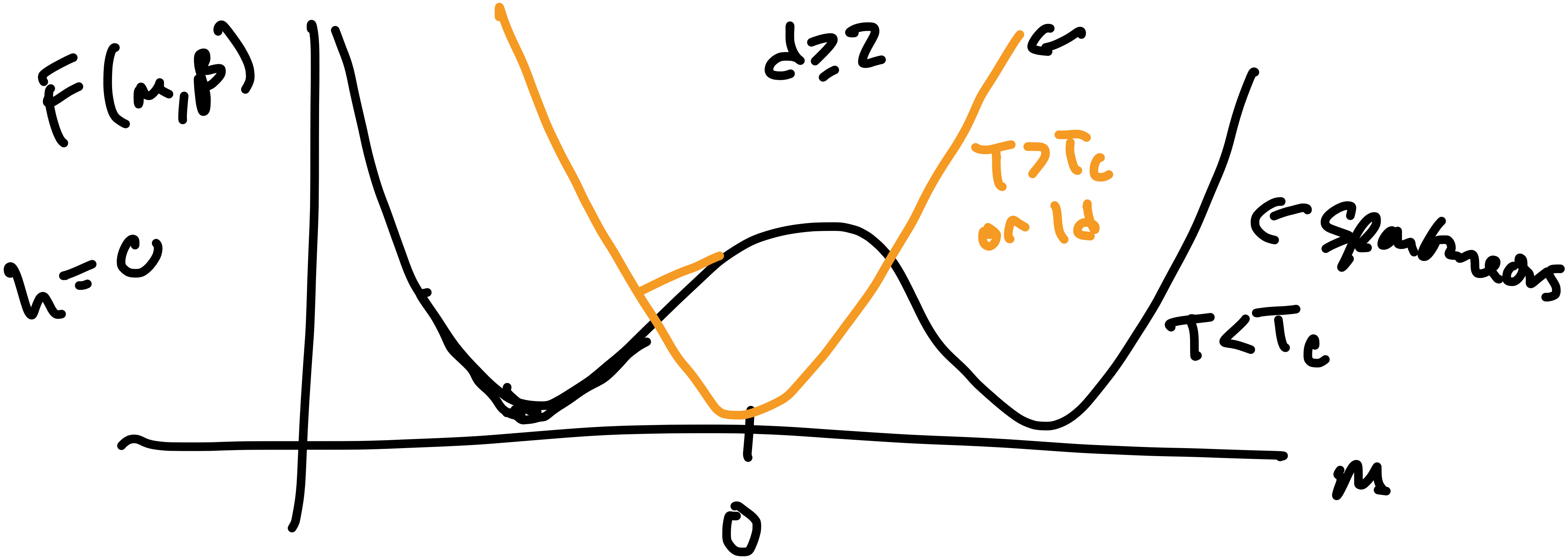


PBCs $\quad s_{N+1} = s_1$

1d: too cheap to flip a spin

$\Rightarrow$ no zero field spontaneous mag. if $T > 0$

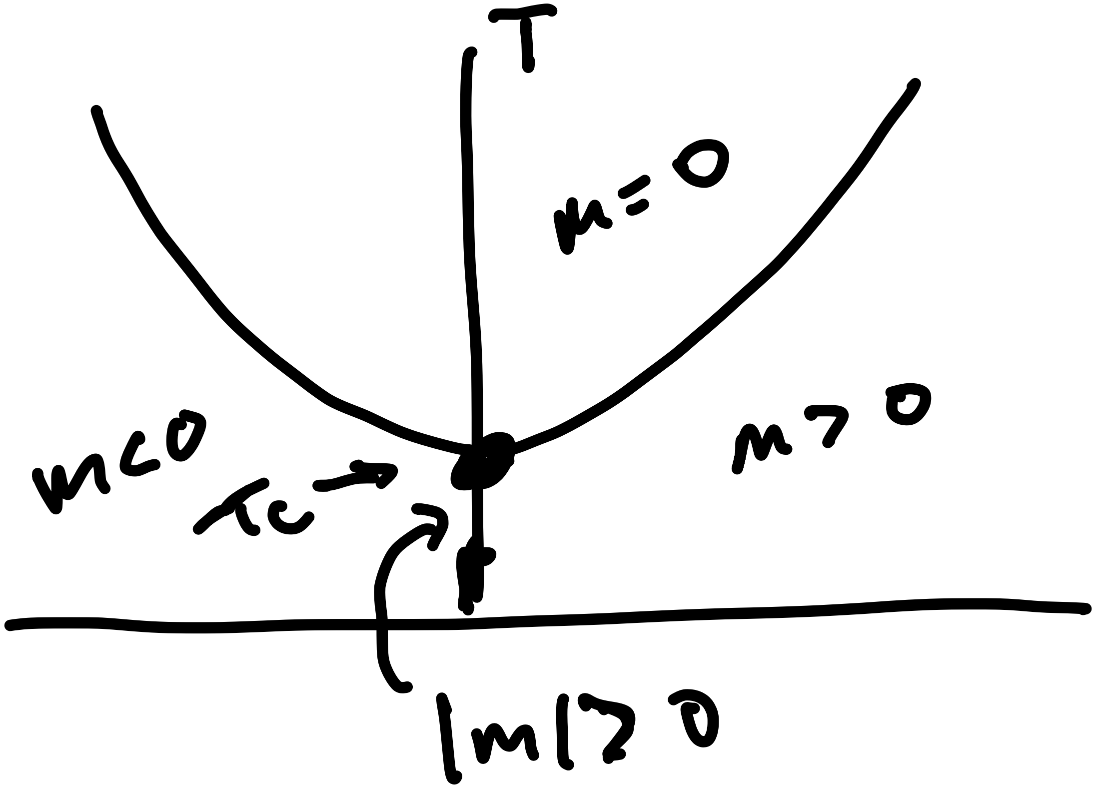
2d and higher – spontaneous mag.

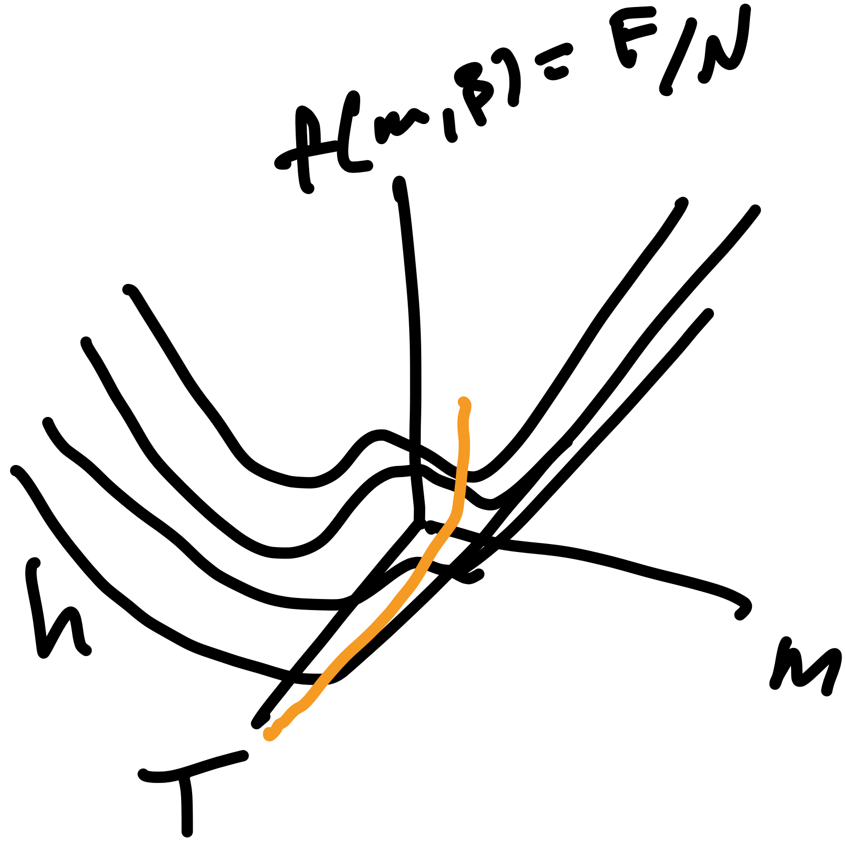


$$m = \frac{1}{N} \sum s_i$$

near t_c

$$f(m) = a + bm^2 + cm^3$$


T
m = 0
m < 0
m > 0
Tc
|m| > 0
h


f(m, β) = F/N
M
T

$$\mathcal{H}_{1d} = \sum_{i=1}^{N} -J s_i s_{i+1} - \sum_{i=1}^{N} h s_i$$

$$Z = \sum_{\{\text{states}\}} e^{-\beta \mathcal{H}(\text{state})}$$

$$\text{"}\vec{x}\text{"} = (s_1, s_2, s_3, \dots, s_N)$$

$$\#\text{states} = 2^N$$

$$= \sum_{s_1 = \pm 1} \sum_{s_2 = \pm 1} \cdots \sum_{s_N = \pm 1} e^{-\beta \mathcal{H}(s_1, s_2, \dots, s_N)}$$

$J = 0$ ✓

States

s_1	s_2	Energy
1	1	$-2h$
1	-1	0
-1	1	0
-1	-1	$2h$

$$\sum_{s_1} \sum_{s_2} e^{-\beta \sum_{i=1}^{2} -h s_i}$$

$$= \sum_{s_1} \sum_{s_2} e^{+\beta h (s_1 + s_2)}$$

$$= \sum_{s_1} \left[e^{\beta h (s_1 + 1)} + e^{\beta h (s_1 - 1)} \right.$$

$$\rightarrow = e^{\beta h(-1+1)} + e^{\beta h(-1-1)} + e^{\beta h(1+1)} + e^{\beta h(1-1)}$$

$$Z = \sum_{s_1} \sum_{s_2} \cdots \sum_{s_N} e^{\beta h \sum_{i=1}^{N} s_i}$$

$$= \left(\sum_{s=\pm 1} e^{\beta h s} \right)^N = \left(e^{-\beta h} + e^{+\beta h} \right)^N$$

$$= \left[2\cosh(\beta h) \right]^N$$

$$= e^{-N\beta h} \left(1 + e^{\underbrace{2\beta h}_{t}} \right)^N$$

for $h=0$

$$Z = \sum_{\text{states}} e^{\beta J \sum_{i=1}^{N} s_i s_{i+1}}$$

s_i $\uparrow\uparrow\downarrow\downarrow\uparrow\downarrow\uparrow\uparrow$

$\tilde{s}_i$ $\uparrow\downarrow\uparrow\downarrow\downarrow\downarrow\uparrow \cdots$

$$= \sum_{\text{states}} e^{\beta J s_1 s_2} e^{\beta J s_2 s_3} \cdots e^{\beta J s_N s_1}$$

$s_1 s_2 = \pm 1 \quad (\times 2)$

$\pm 1 \quad \pm 1$

$\tilde{s}_i = s_i s_{i+1}$

Defect variable

$$Z = \lambda \sum_{\text{states } \tilde{s}_i} e^{\beta J \tilde{s}_i}$$

$$Z = 2 \sum_{\tilde{S}_1 = \pm} \sum_{\tilde{S}_2 = \pm 1} \cdots \sum_{\tilde{S}_N = \pm 1} e^{\beta J \sum_{i=1}^{\tilde{N}} \tilde{S}_i}$$

$$= 2 \left(e^{\beta J} + e^{-\beta J} \right)^N$$

$h = 0$

$|J| > 0$

$$= 2 \left(2 \cosh (\beta J) \right)^N$$

$$Z_{1d} = \sum_{\text{states}} e^{+\beta J \sum_{i=1}^{N} s_i s_{i+1} + \beta h \sum_{i=1}^{N} s_i}$$

$$M(\beta, h, J) = \left\langle \sum_{i=1}^{N} s_i \right\rangle = k_B T \frac{\partial \ln Z}{\partial h}$$

Spontaneous magnitization?

$$\lim_{h\to 0} |M(h,\beta,J)| > 0 \ ?$$

$$F = -k_B T \ln Z$$

$$\ln Z = -\beta F$$

$$M = k_B T \frac{\partial \ln Z}{\partial h} = k_B T \frac{\partial}{\partial h}[-\beta F]$$

$$= -\left(\frac{\partial}{\partial h} F\right)_{T,N,J}$$

$$m = -\frac{\partial}{\partial h} f$$

$$Z = \sum_{\text{states}} e^{\beta J \sum_{i=1}^{N} s_i s_{i+1} + \beta h \sum s_i}$$

$$\rightarrow \quad |1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$|-1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$P \leftarrow$ matrix, 2×2

$$P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}$$

$$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad v_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad a v_1 + b v_2$$

$$P_{11} = v_1^T (P) v_1 = (1\,0) \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= (1\,0) \begin{pmatrix} P_{11} \\ P_{21} \end{pmatrix} = P_{11}$$

$$P_{s_i s_{i+1}} = \langle s_i | P | s_{i+1} \rangle$$

$$Z = \sum_{\text{states}} e^{\beta J \sum_{i=1}^{N} s_i s_{i+1} + \beta h \sum s_i}$$

$$e^{\beta J s_1 s_2 + \beta h s_1}$$

$$= \sum_{s_1} \sum_{s_2} \cdots \sum_{s_N} \langle s_1 | P | s_2 \rangle \langle s_2 | P | s_3 \rangle \cdots \langle s_N | P | s_1 \rangle$$

$$\sum_{s_i = \pm 1} |s_i\rangle\langle s_i| = \text{“}1\text{”} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \sum_{s_1} \langle s_1 | P^{I} P^{I} P^{I} P^{I} \cdots P | s_1 \rangle$$

$$= \mathrm{Tr}(P^N)$$

$\mathrm{Tr}(P^N)$

λ_1, λ_2

eigenvalues of P

$$P = U^{-1} D U$$

$$D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

$$P \cdot P \cdot P \cdot \ldots P = U^{-1} D U U^{-1} D \ldots U^{-1} D U$$

$$= U^{-1} D^N U$$

$$\mathrm{Tr}(ABC) = \mathrm{Tr}(CAB) = \mathrm{Tr}(BCA)$$

$$Z = \mathrm{Tr}(P^N) = \mathrm{Tr}(D^N) = \lambda_1^N + \lambda_N^N$$

turns out

$$P_{s_1 s_2} = e^{\beta J s_1 s_2 + h(s_1+s_2)/2}$$

$$\mathcal{H} = -\sum_{i=1}^{N} J s_i s_{i+1} - \frac{h}{2} \sum_{i=1}^{N} (s_i + s_{i+1})$$

$$= \sum_{i=1}^{N} \left(-J s_i s_{i+1} - \frac{h}{2}(s_i + s_{i+1}) \right]$$

$$= \sum_{i=1}^{N} \mathcal{H}_i$$