Math techniques

derivatives

$$\frac{d}{dx}(ab^n) = nab^{n-1}$$

product & quotient rule

$$\frac{d}{dx}\left(f(x)\,g(x)\right) = f(x)\frac{dg(x)}{dx} + g(x)\frac{df(x)}{dx}$$

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{d}{dx}\left[f(x)\,g(x)^{-1}\right]$$

$$= g^{-1}f' + -1/g^2 f = \frac{gf' - fg'}{g^2}$$

Chain rule

$f(g(x))$

$$\frac{d\, f(g(x))}{dx} = \left[\frac{d}{dx} g(x)\right] f'(g(x))$$

$$f(g(x)) = e^{-\beta \mathcal{H}(x)} \iff f(x) = e^{x} \quad g(x) = -\beta \mathcal{H}(x)$$

$$\frac{df}{dx} = \left(-\beta \frac{d}{dx} \mathcal{H}(x)\right) \cdot e^{-\beta \mathcal{H}(x)}$$

$$\frac{d}{dx} f(q_1, q_2, \ldots, q_N) = \sum_{i=1}^{N} \frac{df}{dq_i} \frac{dq_i}{dx}$$

Integrals

$$\frac{d}{dx}(fg) = f'g + g'f$$

$$fg = \int dx \frac{df}{dx} g(x) + \int dx \frac{dg}{dx} f(x)$$

$$= \int g(x) df + \int f(x) dg$$

$$\int u\,dv = uv - \int v\,du$$

$$\int_a^b u\,dv = (uv)\Big|_a^b - \int_{v=a}^{v=b} v\,du$$

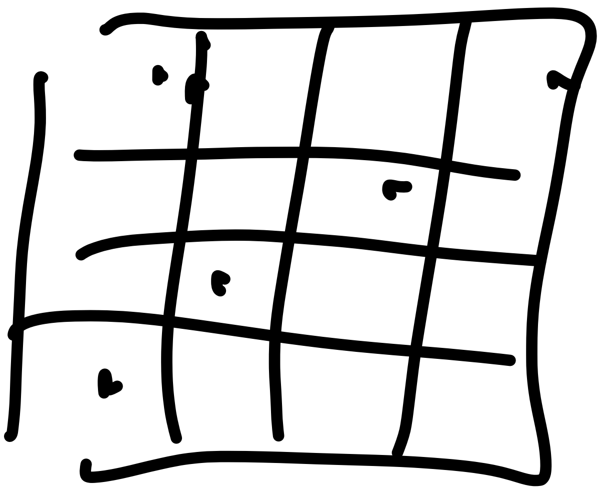
$$\int_{-\infty}^{\infty} \delta(x-a) f(x)\, dx = f(a)$$

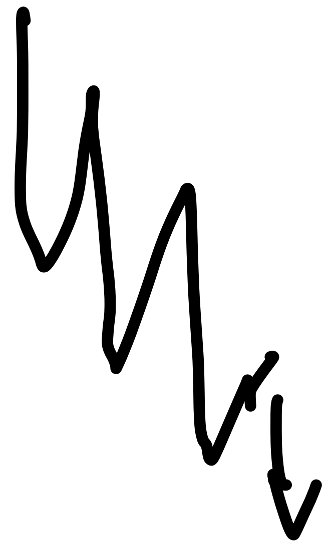
$$\int_{-\infty}^{\infty} dx\, e^{-(x-\mu)^2/2\sigma^2} = \sqrt{2\pi\sigma^2}$$

$$\int dx\, x^n = \frac{1}{n+1} x^{n+1}$$

Counting:

assign n things to N categories or spots





$$N \cdot (N-1) \cdots (N-n)$$

$$\parallel$$

$$\frac{N!}{(N-n!)}\, n!$$

$$\binom{N}{n} = \frac{N!}{(N-n)!\, n!}$$

$$\log_b(b^x) = x$$

$$\ln(b^x) = x\ln(b)$$

$$\ln(xy) = \ln(x) + \ln(y)$$

$$\frac{d}{dx}\ln(x) = \frac{1}{x}$$

$$e^{\ln(x)} = x$$

$$e^{x+y} = e^x e^y$$

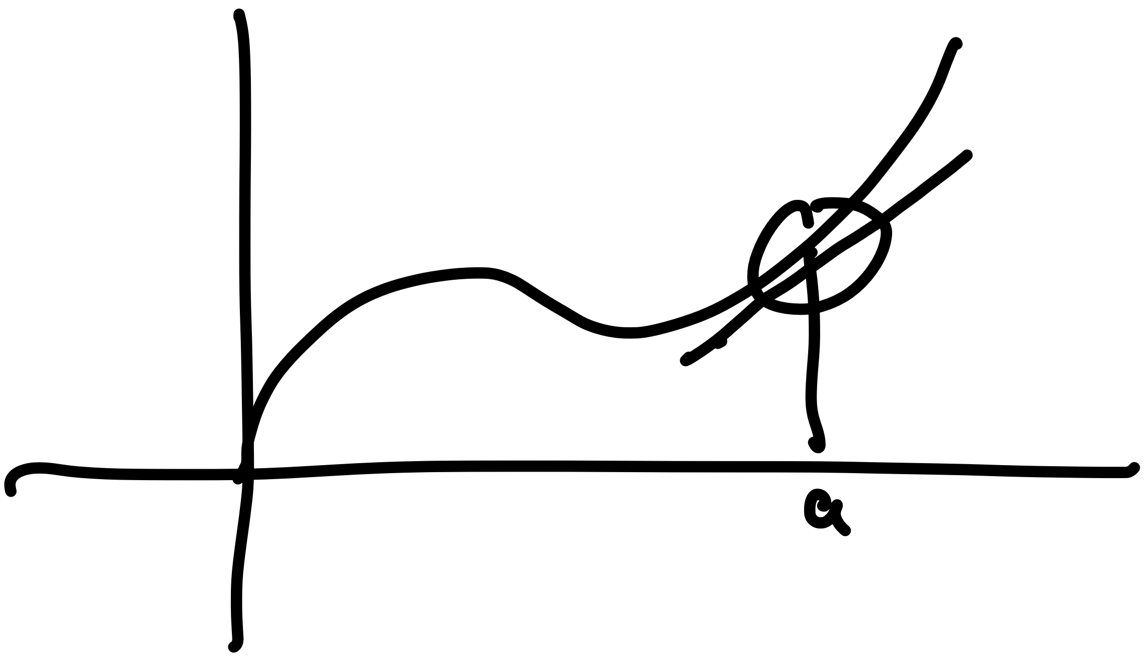
$$\ln(N!) \approx N\ln N - N$$

$$\Leftrightarrow N! \approx N^N e^{-N}$$

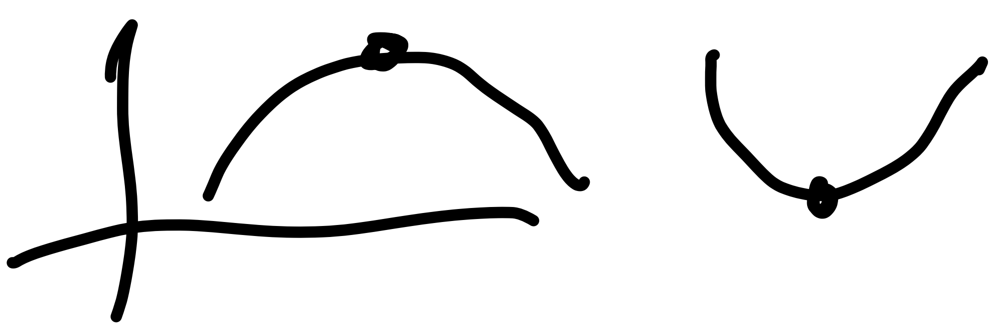
Taylor series

$$f(x) \approx f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2} f''(a) + \dots$$

ith deriv



$$f(x) = \sum_{i=0}^{\infty} \frac{(x-a)^i f^{(i)}(a)}{i!}$$



$$e^x = e^0 + (x-0)\frac{de^x}{dx}\Big|_0 + \frac{(x-0)^2}{2}\frac{d^2e^x}{dx^2}\Big|_0 + \cdots$$

$$= 1 + x + x^2/2 + x^3/3! + \cdots$$

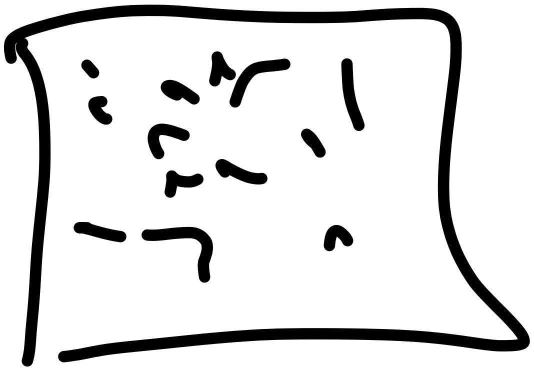
$$= \sum_{n=0}^{\infty} x^n/n!$$

$$\int \sin(x)\, e^{\lambda x}$$

$$e^{\lambda x} \approx 1 + \lambda x + \frac{\lambda^2 x^2}{2}$$

$$\int \sin(x)\,dx + \int \sin(x)\, x\, dx$$

Microcanonical ensemble



$F = ma$

$\Rightarrow$ const E

N, V, E closed isolated

S driving force for processes, maximized at equilibrium

$\hookrightarrow$ # states of system is maximized

$$S = k_B \ln \Omega$$

2 bodies in contact
heat flows until $\frac{1}{T_1} = \frac{1}{T_2}$

$$dE = \bar{d}q + \bar{d}w$$

$$\frac{1}{T} dE = \underbrace{\frac{1}{T}\bar{d}q}_{dS} + \underbrace{\frac{1}{T}\bar{d}w}_{\frac{1}{T}(-PdV + \mu dN)}$$

$$S(N, V, E)$$

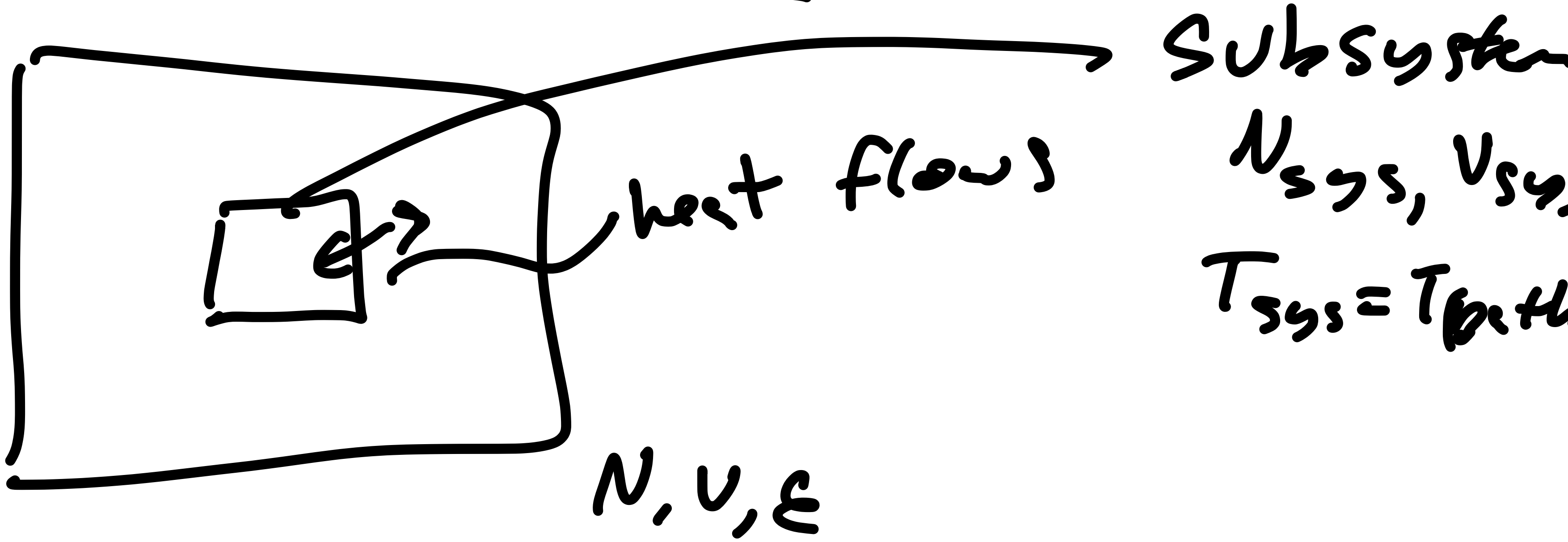
$$dS = \left(\frac{\partial S}{\partial N}\right)_{V,E} dN + \left(\frac{\partial S}{\partial V}\right)_{E,N} dV + \left(\frac{\partial S}{\partial E}\right)_{N,V} dE$$

In micro

$$dS = -\frac{\mu}{T} dN + \frac{P}{T} dV + \frac{1}{T} dE$$

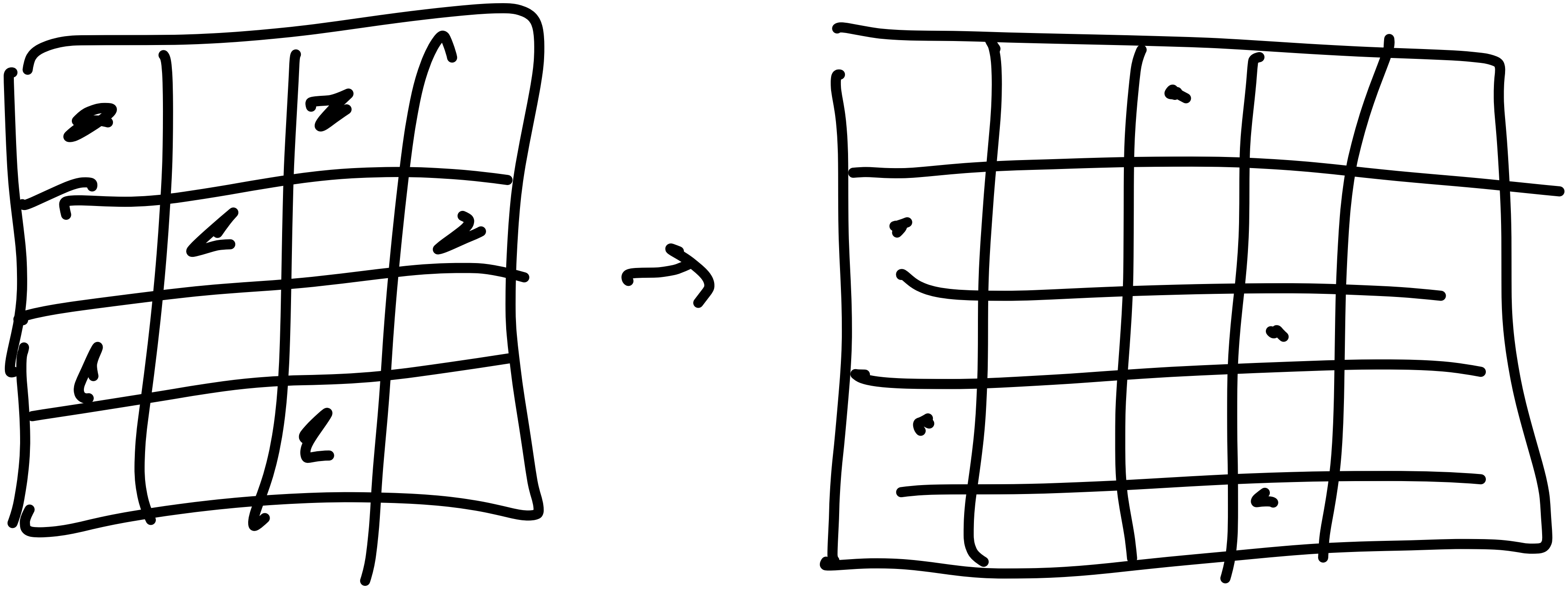
$$\partial S/\partial N = -\mu/T \qquad \partial S/\partial V = P/T \qquad \partial S/\partial E = 1/T$$

Canonical ensemble



Expanded entropy around
state all energy is in bath
system is small

\# States where $\boxed{\text{System}}$ (System + bath) has cfg X
$\propto e^{-\mathcal{H}(X)/k_B T}$



$$Z = \sum \text{number of states}$$

$$= \sum_{i=1}^{N_{states}} e^{-\varepsilon_i / k_B T}$$

$$\text{or} \quad = \int d\vec{x}\, e^{-\mathcal{H}(\vec{x})/k_B T}$$

$$P(x) = e^{-\mathcal{H}(x)/k_B T} / Z$$

$$P(\text{state } i) = e^{-\varepsilon_i / k_B T} / Z$$

$$P(\text{state } i) = e^{-\varepsilon_i/k_BT} \Big/ \sum e^{-\varepsilon_i/k_BT}$$

$$Z = \sum_{i=1}^{N_{\text{states}}} e^{-\mathcal{H}[\text{state } i]/k_BT}$$

$a \times \varepsilon_1 \quad b \times \varepsilon_2 \quad c \times \varepsilon_3$

$$= \sum \Omega(\varepsilon)\, e^{-\varepsilon_i/k_BT}$$

Energy levels

$\binom{N}{N_{up}} \quad -mg(2N - N_t) = \varepsilon$

↑	↓	↑

site 1 2 3

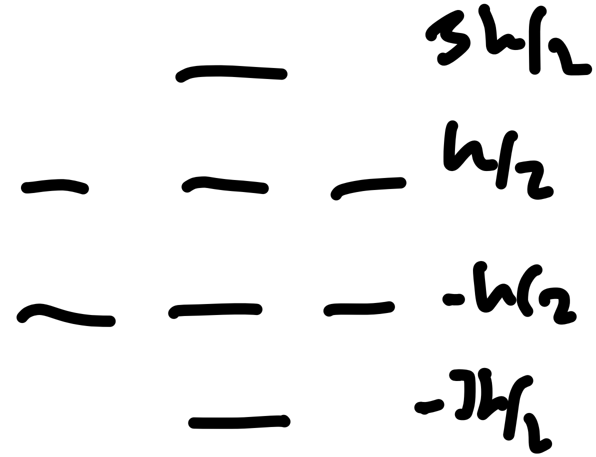
s_i s_1 s_2 s_3

$$\mathcal{E} = \sum_{\text{spin } i} h s_i$$

$$s_i = + \tfrac{1}{2}, \; -\tfrac{1}{2}$$

$3h/2$	↑↑↑	↓↑↑	$h/2$
$h/2$	↑↑↓	↓↑↓	$-h/2$
$h/2$	↑↓↑	↓↓↑	$-h/2$
$-h/2$	↑↓↓	↓↓↓	$-3h/2$

Energy diagram



1 3 3 1

$\binom{3}{0}$ $\binom{3}{1}$ $\binom{3}{2}$ $\binom{3}{3}$

total # states is $2^{N \text{ sites}}$

$$Z = \sum_{\text{states}} e^{-\beta h \sum s_i} \qquad E_{\text{total}} = -\frac{h}{2} N_{\text{down}} + h/2\, N_{\text{up}}$$

$$= \sum_{s_1=-1/2}^{1/2} \sum_{s_2=-1/2}^{1/2} \sum_{s_3=-1/2}^{1/2} \cdots \sum_{s_N=-1/2}^{1/2} e^{-\beta h \sum_{i=1}^{N} s_i}$$

$$= \left(\sum_{s_i} e^{-\beta h s_i} \right)^N = \left(e^{-\beta h/2} + e^{\beta h/2} \right)^{N_{\text{sites}}}$$

$$Z = \sum_{\text{states}} e^{-\beta E(\text{state})} = e^{-\frac{3h}{2}\beta} + e^{-} + \cdots e^{+\frac{3h}{2}\beta}$$

$$Z = \sum_{n_{\text{down}}} \binom{N}{n_{\text{down}}} e^{-\beta h/2 (2N - n_{\text{down}})}$$

$N = 3$

$$Z = \sum_{S_1=-1/2}^{1/2} \sum_{S_2=-1/2}^{1/2} \sum_{S_3=-1/2}^{1/2} e^{-\beta h[S_1 + S_2 + S_3]}$$

$3/2 \quad 1/2$

$-1/2 \quad -3/2$

$$Z = \left(e^{-\beta h/2} + e^{\beta h/2}\right)^N$$

$$E = -\frac{\partial \ln Z}{\partial \beta} = N \cdot \frac{1}{e^{-\beta h/2} + e^{\beta h/2}} \cdot \left[-\beta h/2 \, e^{-\beta h/2} + \beta h/2 \, e^{\beta h/2}\right]$$