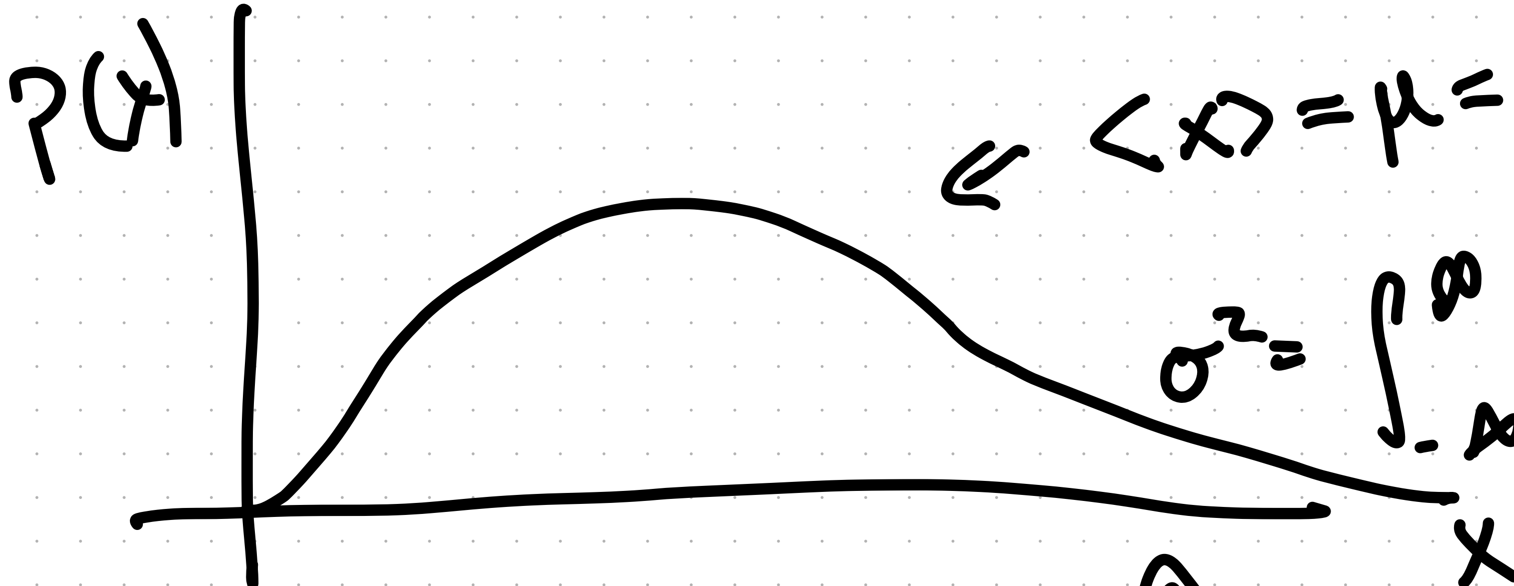
Q:

$$\underset{\sim}{f(x)} = \int_a^x g(x)\,dx \qquad C_{AB}(t) = \int_0^t \underset{\sim}{A(t)B(0)}\,dt$$

$$\langle f(x) \rangle = \int_{-\infty}^{\infty} f(x)P(x)\,dx$$

$$\mathrm{Var}\, f(x) = \int_{-\infty}^{\infty} \left(f(x) - \langle f(x) \rangle\right)^2 P(x)\,dx$$

Lecture 3



$$\Leftarrow \langle x \rangle = \mu = \int_{-\infty}^{\infty} x P(x)\, dx$$

$$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 P(x)\, dx$$

$$x_1, x_2, \ldots, x_N$$

sampled from this

$$\mu_N = \frac{1}{N} \sum_{i=1}^{N} x_i$$

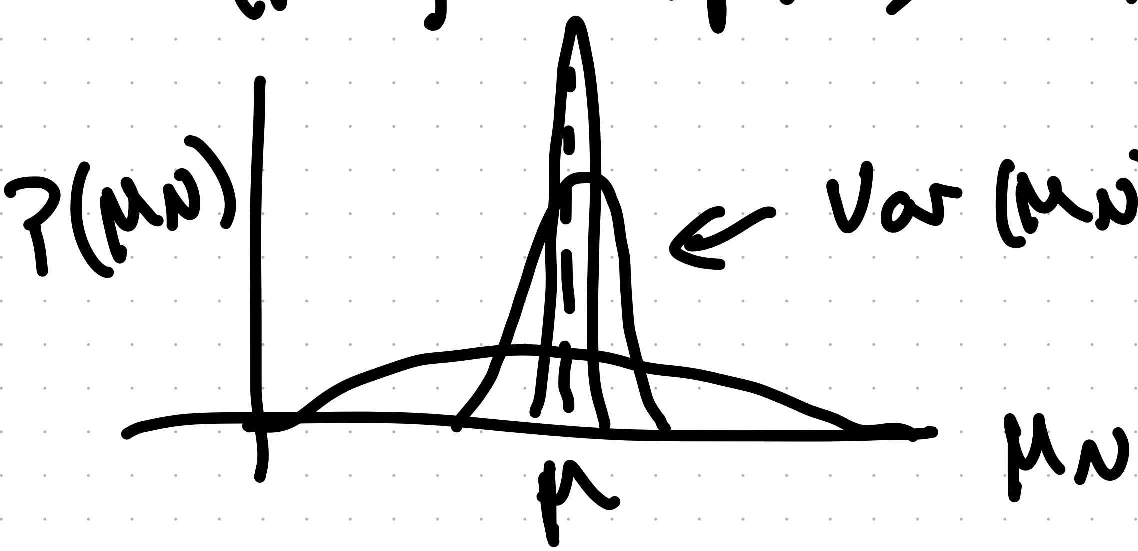
how close is μ_N to μ

$$\mu_N = \frac{1}{N}\sum_{i=1}^{N} x_i$$

x_i are independent samples

$$\langle \mu_N \rangle = \frac{1}{N}\sum_{i=1}^{N} \langle x_i \rangle = \frac{1}{N}\sum_{i=1}^{N} \mu = \frac{N}{N}\mu = \mu$$

$$\mathrm{Var}(\mu_N) = \langle \mu_N^2 \rangle - \langle \mu_N \rangle^2$$



$$\mathrm{Var}(\mu_N) = \underbrace{\langle \mu_N^2 \rangle}_{?} - \underbrace{\langle \mu_N \rangle^2}_{\sim \mu^2}$$

$$\mu_N = \frac{1}{N} \sum_{i=1}^{N} x_i$$

$$\langle \mu_N^2 \rangle = \left\langle \left(\frac{1}{N} \sum_{i=1}^{N} x_i \right) \left(\frac{1}{N} \sum_{j=1}^{N} x_j \right) \right\rangle$$

$$= \left\langle \frac{1}{N^2} \sum_{i,j=1}^{N} x_i , x_j \right\rangle = \frac{1}{N^2} \sum_{i,j} \langle x_i x_j \rangle$$

$$\langle x_i x_j \rangle = \begin{cases} \langle x_i \rangle^2 & i = j \\ \underbrace{\langle x_i \rangle \langle x_j \rangle}_{\mu^2} & i \neq j \end{cases} \leftarrow \text{independent}$$

$$\langle \mu_N \rangle = \frac{1}{N^2} \sum_{i,j} \langle x_i x_j \rangle \overset{?}{=} \frac{1}{N^2} \sum_{i=1}^{N} \underbrace{\sum_{j=1}^{N}} \langle x_i x_j \rangle$$

$$= \frac{1}{N^2} \sum_{i=1}^{N} \left(\langle x_i \rangle^2 + \underbrace{(N-1)\mu^2}_{N\mu^2 - \mu^2} \right)$$

$$= \frac{1}{N^2} \sum_{i=1}^{N} \left(\underbrace{\langle x_i \rangle^2 - \mu^2}_{\mathrm{Var}(x_i)} \right) + \frac{1}{N^2} \sum_{i=1}^{N} N\mu^2 = \mu^2$$

$$\mathrm{Var}(\mu_N) = \langle \mu_N^2 \rangle - \mu^2$$

$$= \left[\left(\frac{1}{N^2} \sum_{i=1}^{N} \mathrm{Var}(x_i)\right) + \cancel{\mu^2}\right] - \cancel{\mu^2}$$

$$\sigma_N^2 = \frac{1}{N^2} \sum_{i=1}^{N} \sigma^2 = \frac{1}{N} \sigma^2$$

$$\boxed{\sigma_N = \sigma / \sqrt{N}}$$

"Standard error of the mean"

$\mu_N \to \mu$

like $\frac{1}{\sqrt{N}}$

Central Limit Theorem

x_i points sampled from $P(x)$

$$\mu_N = \frac{1}{N}\sum_{i=1}^{N} x_i$$

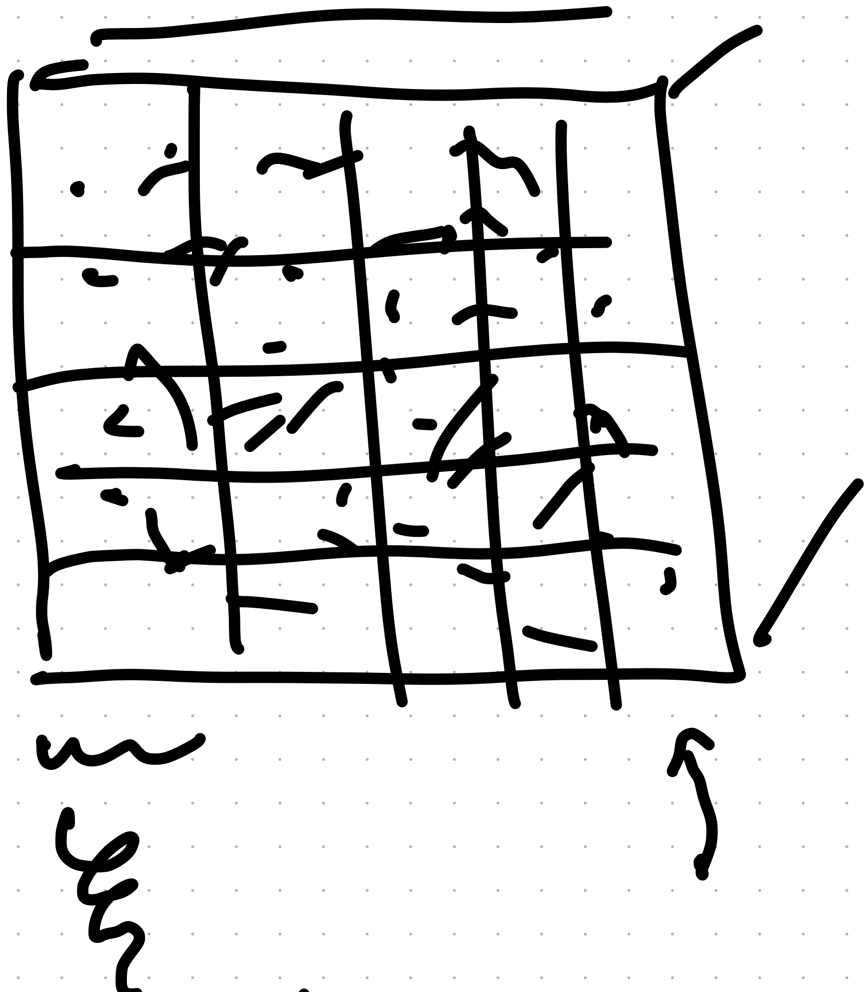
mean zero

var

$$P(\mu_N - \mu) \xrightarrow[N\to\infty]{} \mathcal{N}(\underline{0}, \sigma^2/N)$$

$$= \frac{1}{A} e^{-\frac{(\mu_N-\mu)^2}{2(\sigma^2/N)}}$$

Statistical Mechanics



$$N_{boxes} = \frac{V}{\xi^d}$$

Calculate A_i in each box

$\langle A_i \rangle = \langle A \rangle$ ~~for~~ system

$$\mathrm{Var}\, A_i \sim \frac{1}{\sqrt{N_{boxes}}}$$

Classical Mechanics

$$\vec{r} = (\vec{r}_1, \ldots, \vec{r}_N) \qquad \vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$$

$$\vec{v} = (\vec{v}_1, \ldots, \vec{v}_N) \qquad \vec{a} = \dot{\vec{v}} = \ddot{\vec{r}}$$

Newton's Equation says $F = ma$

N differential equations

$$m_i \ddot{\vec{r}}_i = \vec{F}_i(\vec{r}_1 \ldots \vec{r}_N)$$

if we know $\vec{r}(0)$, $\vec{v}(0)$ and $F(\vec{r})$ then everything is determined

If no friction, and know

$U(\vec{r})$

$$\vec{F}(\vec{r}) = -\nabla U(\vec{r}) \text{ ie}$$

$$F_i(\vec{r}) = -\frac{\partial U(\vec{r})}{\partial \vec{r}_i} \quad \text{(no veloc.)}$$

$$\underline{\nabla} A = \left(\frac{\partial A}{\partial \vec{r}_1}, \frac{\partial A}{\partial \vec{r}_2}, \dots, \frac{\partial A}{\partial \vec{r}_N} \right)$$

$$\mathcal{E}(\vec{r},\vec{v}) = \underbrace{\sum_{i=1}^{N} \frac{1}{2} m v_i^2}_{\text{kinetic}} + \underbrace{U(r)}_{\text{potential}}$$

momenta $\vec{p}_i = m_i \vec{v}_i$

If $F = -\nabla U$, forces are "conservative"

total energy is conserved

$$\mathcal{E} = \frac{1}{2}mv^2 + U(r) = \frac{p^2}{2m} + U(r)$$

$$\frac{d\mathcal{E}}{dt} = \frac{1}{2}m\left(v\dot{v} + \dot{v}v\right) + \frac{dU(r)}{dt}$$

$$= mva + \frac{dU(r)}{dt}$$

$$= vF + \frac{dU(r)}{dt} = vF + \overbrace{\left(\frac{\partial U}{\partial r}\right)}^{-F}\overbrace{\frac{\partial r}{\partial t}}^{v} = 0$$

Chain rule $\quad \frac{dX}{dt} = \sum_{i=1}^{N}\left(\frac{\partial X}{\partial l_i}\right)\frac{\partial l_i}{\partial t} = \sum_{i=1}^{N}\frac{\partial X}{\partial l_i}\dot{l}_i$

$$\frac{dE}{dt} = \vec{v} \cdot \vec{F} - (\nabla U) \cdot \dot{\vec{r}}$$

in higher dimension

$$= \vec{v} \cdot \vec{F} - \vec{F} \cdot \vec{v} = 0$$

Energy is conserved

Lagrangian Mechanics

for conservative systems

$$\mathcal{L}(\vec{r}, \dot{\vec{r}}) = K(\dot{\vec{r}}) - U(\vec{r})$$

Euler - Lagrange Equation

(Sec 1.6) Tuckerm

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}_i}\right) - \frac{\partial \mathcal{L}}{\partial r_i} = 0$$

$$K = \frac{1}{2} m \dot{r}_i^2$$

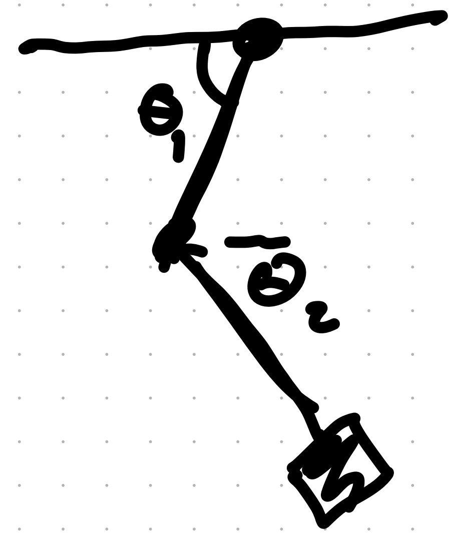
$$\frac{d}{dt}(m\dot{r}_i) = -\frac{\partial U}{\partial r_i}$$

$$\boxed{m\ddot{r}_i = F_i}$$

Why is this helpful?

Other coordinates

$$q_i = f_i(\vec{r})$$



Hamiltonian Mechanics

$\mathcal{H}(\vec{q}, \vec{p})$ in xyz

$\vec{p} = m\dot{\vec{r}}$

In general $p_i = \dfrac{\partial \mathcal{L}}{\partial \dot{q}_i}$

$K(\dot{q}_i) = \frac{1}{2} m \dot{q}_i^2$

$\dfrac{\partial \mathcal{L}}{\partial \dot{q}_i} = m\dot{q}_i = m v_i$
$= p_i$

total energy

$$\mathcal{H}(\vec{q}, \vec{p}) = K + U(q), \quad K = \sum_{i=1}^{N} p_i^2/2m$$

total energy

$$\mathcal{H}(\vec{q},\vec{p}) = K + U(q) \quad , \quad K = \sum_{i=1}^{N} p_i^2/2m$$

$$\boxed{\begin{aligned} \dot{q}_i &= \frac{\partial \mathcal{H}}{\partial p_i} \\ \dot{p}_i &= -\frac{\partial H}{\partial q_i} \end{aligned}}$$

← equations of motion

$\mathcal{H}$ and $\mathcal{L}$ are connected by a "Legendre transform"

[Sec 1.5]

$$\mathcal{H}(\vec{p},\vec{q}) = \sum_{i=1}^{N} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \dot{q}_i - \mathcal{L}$$

$$= \sum_{i=1}^{N} \dot{q}_i p_i - \mathcal{L}$$

$\mathcal{H}$ is total energy

$$\frac{d\mathcal{H}}{dt} = \sum_{i=1}^{N} \left(\frac{\partial \mathcal{H}}{\partial q_i}\right)\left(\frac{\partial q_i}{\partial t}\right) + \left(\frac{\partial \mathcal{H}}{\partial p_i}\right)\left(\frac{\partial p_i}{\partial t}\right)$$

$$\mathcal{H}(\vec{q}, \vec{p})$$

$$= \sum_{i=1}^{N} \left(-\dot{p}_i\right)\dot{q}_i + \left(\dot{q}_i\right)\left(\dot{p}_i\right)$$

$$= \sum_{i=1}^{N} \left[q_i p_i - p_i q_i \right]$$

$$= 0$$