Phase transitions pt 4

Last time:

MF ising model:

spin interacting with average spin of all neighbors

Predicts spontaneous magnetization transition in $h=0$ for all d,

wrong in $d=1$, gets better for higher d

$$C_V \sim |T - T_c|^{-\alpha}$$

$$K_T \sim |T - T_c|^{-\gamma}$$

$$m \sim |T_c - T|^{\beta}$$

Exponents can be the same for many kinds of systems

Universality

Universality behavior:

1) Dimension of the order parameter (n)

magnetization – "M"

scalar, $n=1$

$\rho_L - \rho_g$ – $n=1$

2) Dimension system lives in

liquid gas, 3d ising model, $d=3$

MF ising model:

$$m = \tanh(\beta(2Jmz + h))$$

h kind of like a pressure

$$h = k_B T \underbrace{\tanh^{-1}(m)} - 2mJz$$

$$\tanh^{-1}(m) \underset{m=0}{\approx} m + m^3/3 + \ldots$$

$$h \approx k_B T (m + m^3/3) - 2mJz$$

$$\approx m[k_B T - 2Jz] + \frac{k_B T}{3} m^3$$

$$T_c = \frac{2Jz}{k_B}$$

$$\approx m k_B [T - T_c] + \frac{k_B T}{3} m^3$$

$$P - P_c \sim (\rho - \rho_c)^\delta \operatorname{sign}(\rho - \rho_c)$$

$$h \sim m^\delta \operatorname{sign}(m) \sim m^\delta$$

$\delta = 3$

$$h \approx m k_B [T - T_c] + \frac{k_B T}{3} m^3$$

$$\chi = \frac{\partial m}{\partial h} = 1 / \left(\frac{\partial h}{\partial m} \right)$$

$$\frac{\partial h}{\partial m} \approx k_B (T - T_c) + k_B T m^2$$

$$\lim_{T \to T_c^+} \chi \sim \frac{1}{T - T_c}$$

$m = 0$

$(T - T_c)^{-1}$

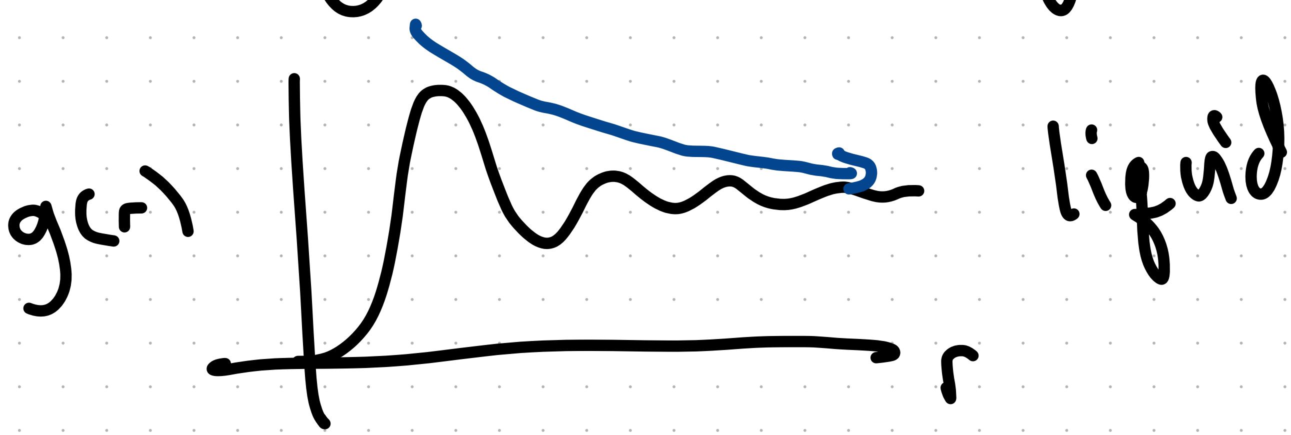
γ

Spatial correlation

Liquids – $g(r)$

how long-ranged is a molecules influence on neighbors

$$\langle \rho(r)\,\rho(r') \rangle \sim g(r-r')$$



$$\delta s_i = s_i - \langle s_i \rangle$$

$$\tilde{C}_{ij} = (s_i - \langle s_i \rangle)(s_j - \langle s_j \rangle)$$

$$C_{ij} = \langle \tilde{C}_{ij} \rangle = \sum_{s_1, s_2, \ldots s_N = \pm 1} C_{ij} \frac{e^{-\beta \mathcal{H}(s_1, \ldots, s_N)}}{Z}$$

↑ spin spin correlation function

$C_{ij} = \langle \delta s_i \delta s_j \rangle$ as a function of $|i-j|$

if a big system

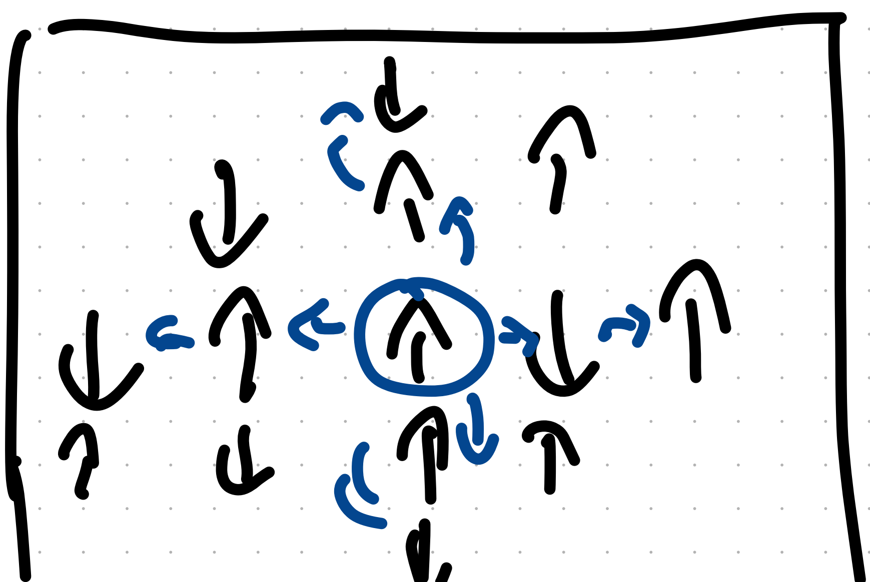
$|i-j|$ large $[r_i - r_j]$

expect spin i spin j are not correlated

$$\langle \delta s_i \delta s_j \rangle \to \langle \delta s_i \rangle \langle \delta s_j \rangle$$
$$= \langle \delta s_i \rangle^2$$

$$\bar{C}_{ij} = \frac{C_{ij}}{Var(s_i)} = \frac{\langle s_i s_j \rangle - \langle s_i \rangle \langle s_j \rangle}{\langle \delta s_i^2 \rangle}$$

ξ length over which things are correlated



Spins correlated with a particular spin

$$= \sum_{j=2}^{N} C_{1j} \approx \int g(r)\, r^2\, dr$$

$$\chi = \frac{1}{N}\langle \delta M^2 \rangle \qquad \delta M = \sum_{i=1}^{N} \delta s_i$$

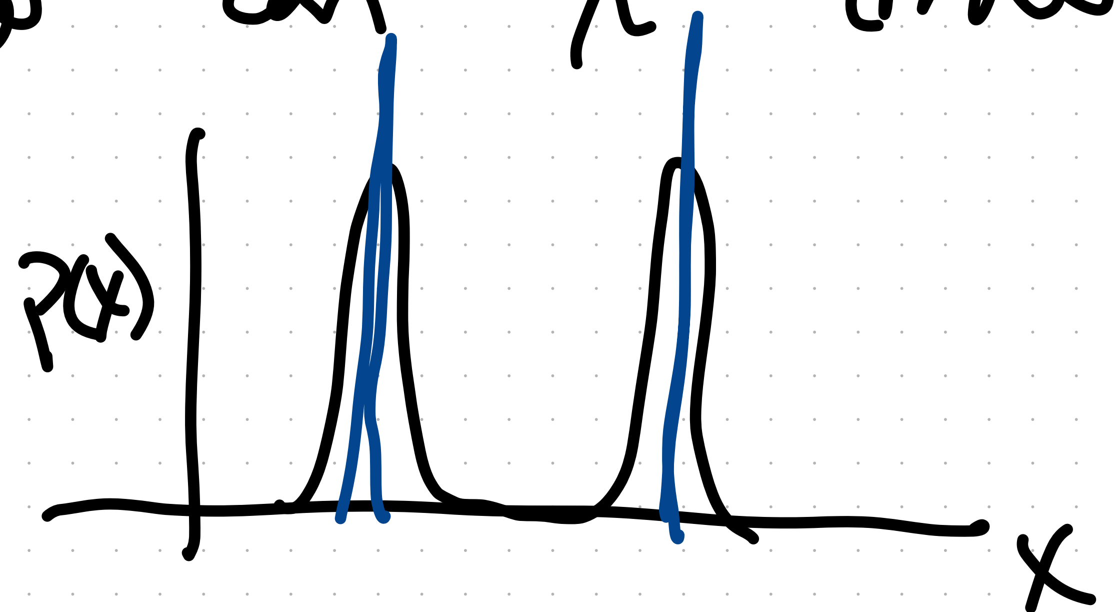
$$= \frac{1}{N} \sum_{i,j} \langle s_i s_j \rangle - \langle s_i \rangle \langle s_j \rangle$$

$$= \frac{1}{N} \cdot N \sum_{j} \langle s_1 s_j \rangle - \langle s_1 \rangle \langle s_j \rangle$$

$$\chi = \sum_j c_{ij} \approx \# \text{ correlated neighb.}$$

How can χ diverge (variance)

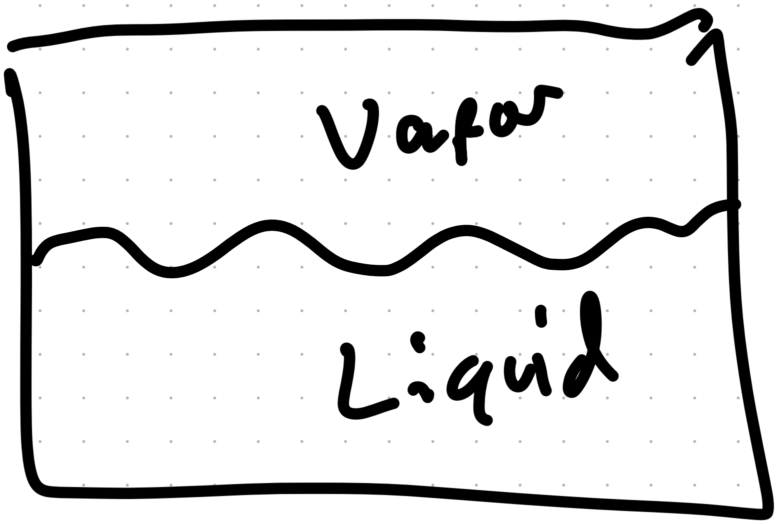
1)



$\mathrm{Var}(X)$

↳ first order phase transition

2 phase coexistence



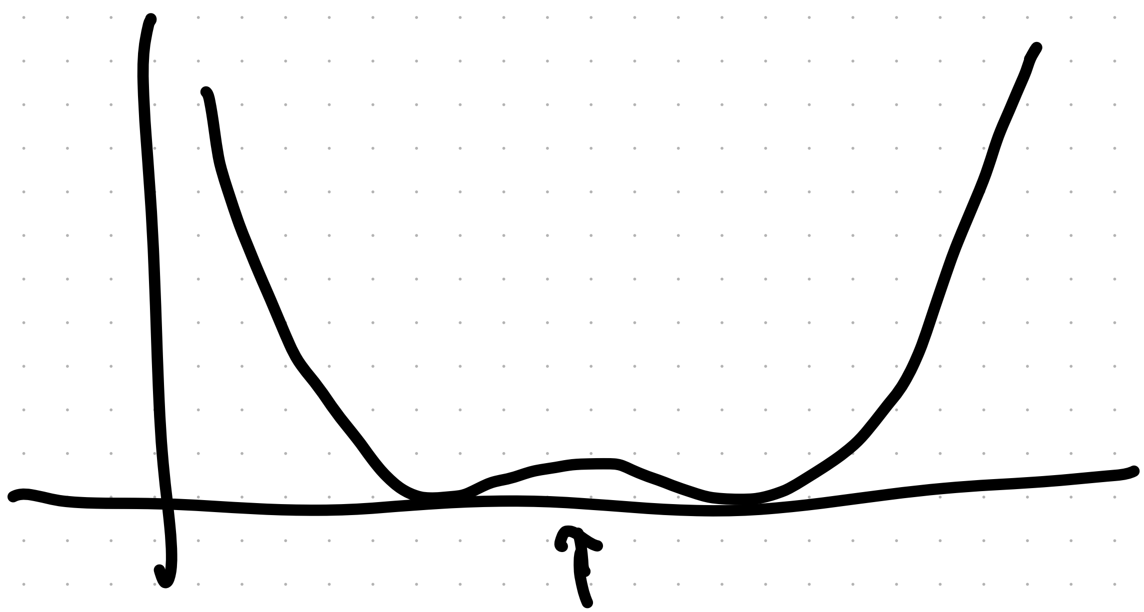
Var(ρ) as N increases

Near critical point – no distinction between the phases

correlations themselves become ∞

$$G(r_{ij}) = C_{ij} \sim \frac{e^{-r/\xi}}{r^{d-2+\eta}}$$

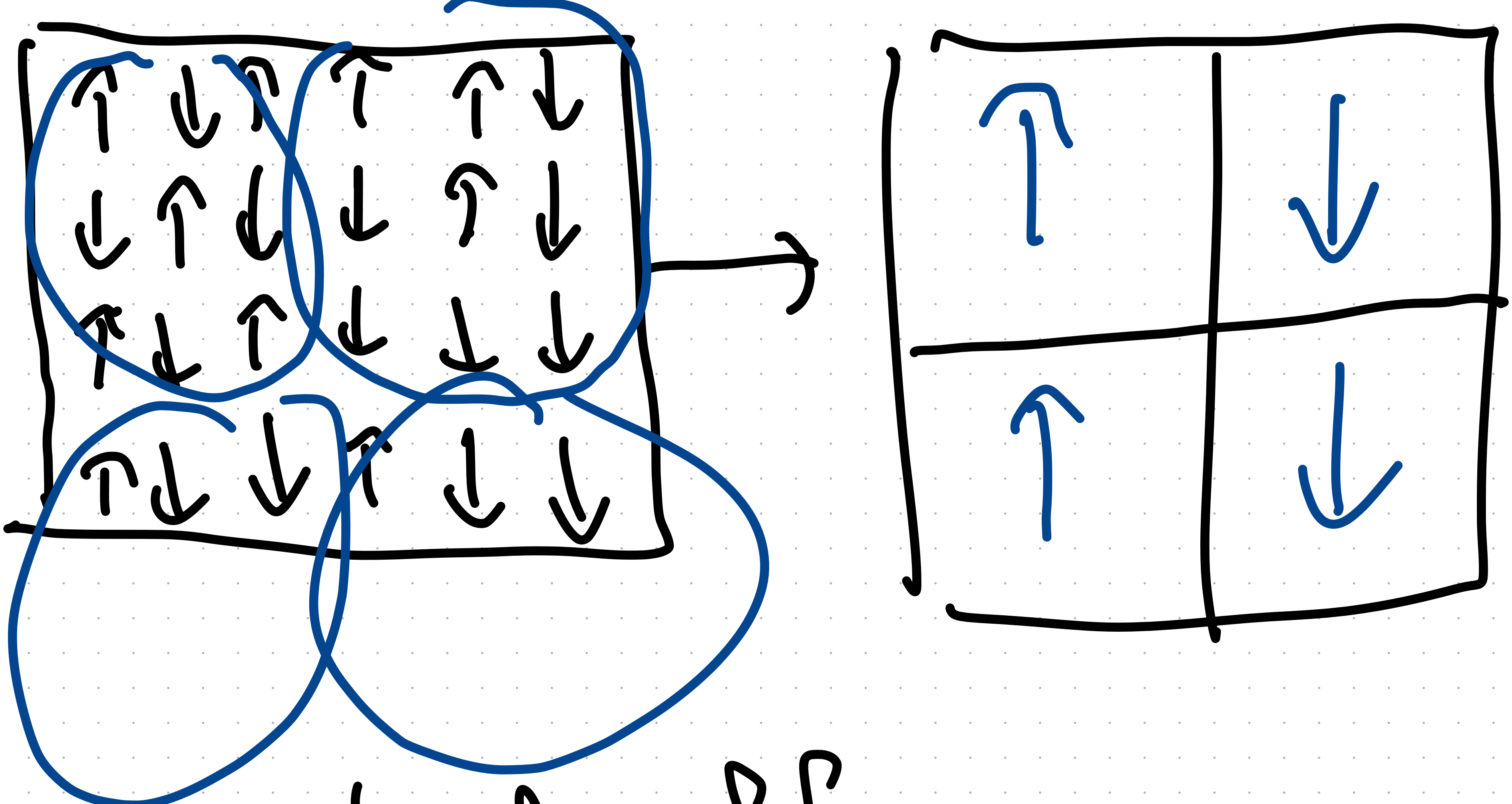
$$\xi \sim |T - T_c|^{-\nu}$$



System looks the same on
small and large length scales
[close to T_c]

Renormalization Group

"coarse grain" system
get back system qualitatively
the same



Kadanoff

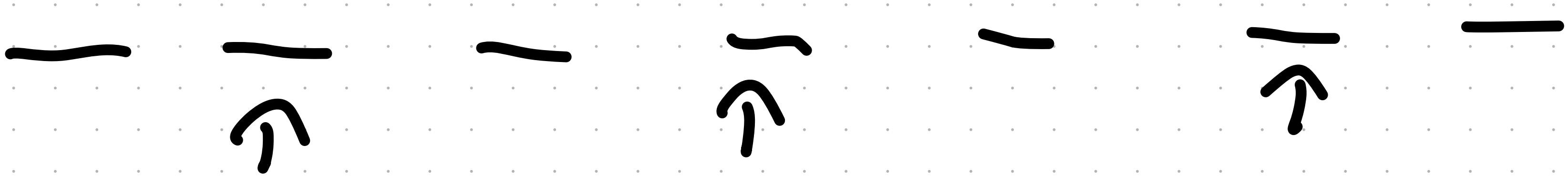
iterate

if $H = \sum_{i,j} J s_i s_j \rightarrow \sum_{i',j'} J' s_i' s_j'$

In 1d:

$$Q(K,N) = \sum_{s_1,\dots s_N = \pm 1} e^{K(s_1 s_2 + s_2 s_3 + \dots s_N s_1)}$$

↑ βJ



$$Q(K,N) = \sum_{s_1, s_3, \dots} e^{K(s_1 + s_3)} + e^{-K(s_1 + s_3)} + e^{K(s_3 + s_5)} + \dots$$

if $e^{k(s+s')} + e^{-k(s+s')} = f(k) e^{k' s s'}$

then $Q(k, N) = f(k)^{N/2} \, Q(k', N/2)$

Kadanoff transformation

$k_0, k_1, k_2 \ldots$

$X \to \to \to \to \to \to K_N \approx \infty$ Renorm. flow

$k_0 = 0$

$K_0 = \infty \to \to \to \to \to \to k_0 \approx 0$

In 2d approximate Kadanoff transformation

$$\underset{k=0}{X} \Leftarrow \leftarrow \leftarrow \leftarrow \underset{K_c}{x} \rightarrow \rightarrow \rightarrow \rightarrow \underset{k=\infty}{X}$$

unstable fixed point

$K_c = 0.50698$

$J/k_B T_c = 0.44069$

[Chandler Ch5
Tuckerman 267]