Phase transitions, Pt 2

Ising model in d-dimensions

$$\mathcal{H} = -\frac{1}{2}\sum_{\langle i,j\rangle} J s_i s_j - h\sum_{i=1}^{N} s_i$$

$s_i = \pm 1$

$\uparrow \; \downarrow \; \uparrow \; \uparrow \; \downarrow$

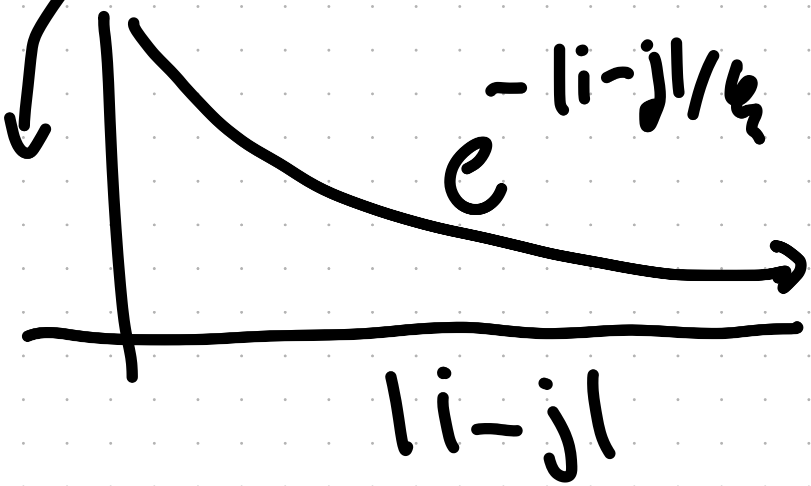
$d = 1$

$$\mathcal{H} = -J\sum_{i=1}^{N} s_i s_{i+1} - h\sum_{i=1}^{N} s_i$$

When is there spontaneous magnetization?

$$\frac{\langle \sum s_i \rangle}{N} = m > 0 \qquad \text{w/ } h = 0 \text{ ?}$$

also: $\langle (s_i - \langle s_i \rangle)(s_j - \langle s_j \rangle) \rangle$ should decay over some distance



$$Z = \sum_{\nu \text{ states}} e^{-\beta E_\nu} \quad , \quad \nu = \{s_1, s_2, \ldots, s_N\}$$

$$= \sum_{s_1=\pm 1} \sum_{s_2=\pm 1} \cdots \sum_{s_N=\pm 1} e^{-\beta \mathcal{H}(s_1, s_2, \ldots, s_N)}$$

$$\mathcal{H}(s_1, \ldots, s_N) = -\sum_{i=1}^{N} \left[J s_i s_{i+1} + h s_i \right]$$

$$\left\langle \sum_{i=1}^{N} s_i \right\rangle = \sum_{\nu} \left(\sum s_i \right) \frac{e^{-\beta E_\nu}}{Z}$$

$$Z = \sum_{s_1, s_2, \ldots, s_N = \pm 1} e^{\beta\left(\sum_{i=1}^{N} J s_i s_{i+1} + h s_i\right)}$$

$$\frac{\partial \ln Z}{\partial h} = \frac{1}{Z}\frac{\partial Z}{\partial h} = \frac{1}{Z}\sum_{\{s_i\}}\left(\beta \sum s_i\right) e^{\beta \sum_{i=1}^{N}\left(J s_i s_{i+1} + h s_i\right)}$$

$$= \beta \left\langle \sum_{i=1}^{N} s_i \right\rangle$$

$$m = \frac{1}{N\beta}\frac{\partial \ln Z}{\partial h} = \frac{1}{N}\left(\frac{\partial \ln Z}{\partial(\beta h)}\right)_{N,T}$$

Spontaneous magnetization

$$\lim_{h \to 0} \frac{k_B T}{N} \left(\frac{\partial \ln Z}{\partial h} \right)_{N,T} = ?$$

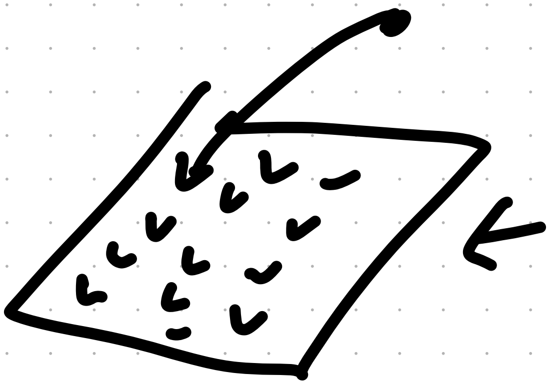
$$\tilde{H} = -\sum_{i=1}^{N} \left(J s_i s_{i+1} + h_i s_i \right)$$

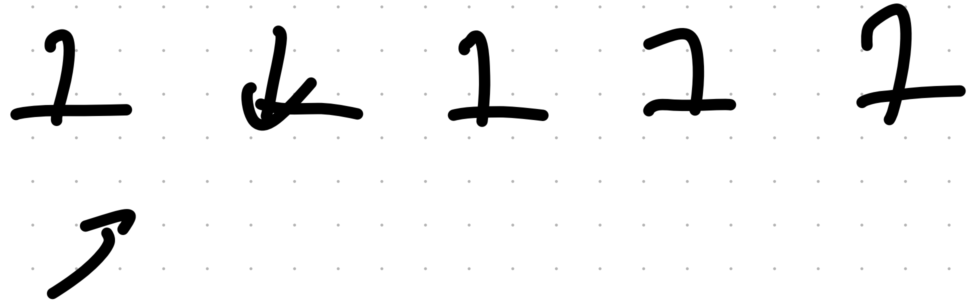
$$\langle s_i s_j \rangle = \frac{\sum s_i s_j e^{-\beta E_\nu}}{Z}$$

$$\langle s_i \rangle = \frac{\partial \ln Z}{\partial (\beta h_i)}$$

$\langle s_i s_j \rangle = ?$ take deriv. $\frac{\partial}{\partial h_j}$

what if $J = 0$





$$\mathcal{H} = -h \sum S_i$$

$$Z = \sum_{\{S_i\}} e^{+\beta h \sum_{i=1}^{N} S_i} = \sum_{S_1} e^{\beta h S_1} \sum_{S_2} e^{\beta h S_2} \cdots \sum_{S_N}$$

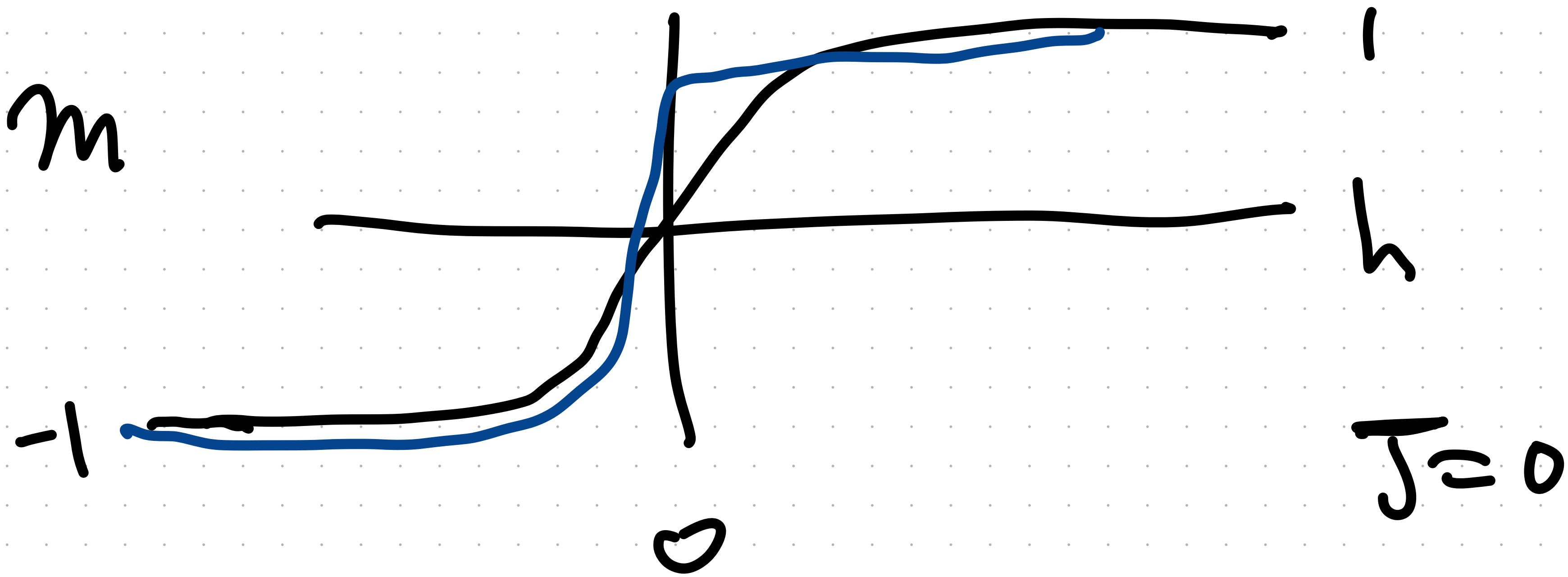
$$= \Big(\sum_{S_1 = \pm 1} e^{+\beta h S_i} \Big)^N = \left(e^{\beta h} + e^{-\beta h} \right)^N$$

$$\frac{\langle \sum S_i \rangle}{N} = \frac{k_B T}{N} \frac{\partial \ln Z}{\partial h}, \quad Z = (e^{\beta h} + e^{-\beta h})^N$$

$$= k_B T \frac{\partial}{\partial h} \ln[e^{\beta h} + e^{-\beta h}]$$

$$= k_B T \cdot \frac{1}{e^{\beta h} + e^{-\beta h}} \cdot \beta [e^{\beta h} - e^{-\beta h}]$$

$$= \frac{e^{\beta h} - e^{-\beta h}}{e^{\beta h} + e^{-\beta h}} = \tanh(\beta h)$$



What about $h=0$?

$$Z = \sum_{\{S_i\}} e^{\beta J \sum_{i=1}^{N} S_i S_{i+1}}$$

$$\tilde{S}_i = S_i S_{i+1}$$

$$S_i S_{i+1} = \begin{cases} +1 & S_i = S_{i+1} \leftarrow 2\text{ ways} \leftarrow \\ -1 & S_i \neq S_{i-1} \leftarrow 2\text{ ways} \leftarrow \end{cases}$$

@ $h=0$

$$Z = 2\sum_{\tilde{S}_i} e^{\sum_{i=1}^{N-1}\beta J\tilde{S}_i}$$

$$= 2\left[\sum_{\tilde{S}_i=\pm 1} e^{\beta J\tilde{S}_i}\right]^{N-1}$$

$$= 2\left[e^{\beta J}+e^{-\beta J}\right]^{N-1}$$

$f = -k_BT \ln Z$

$\mathcal{E} = -\partial \ln Z / \partial\beta$

$m = ?$

$= \dfrac{\partial \ln Z}{\partial \beta h}$

How to solve for $|J| > 0$, $|h| > 0$

Transfer Matrices

$$\mathcal{H} = -\sum_{i=1}^{N} J s_i s_{i+1} - h \sum_{i=1}^{N} s_i$$

$$= -\sum_{i=1}^{N} J s_i s_{i+1} - h \sum_{i=1}^{N} \frac{(s_i + s_{i+1})}{2}$$

$$= \sum_{i=1}^{N} f(s_i, s_{i+1}) \qquad f(s_i, s_{i+1}) =$$

$$-J s_i s_{i+1} - h \frac{s_i + s_{i+1}}{2}$$

$$P_{s,s'} = \begin{pmatrix} e^{-\beta f(1,1)} & e^{-\beta f(1,-1)} \\ e^{-\beta f(-1,1)} & e^{-\beta f(-1,-1)} \end{pmatrix}$$

$$= \begin{pmatrix} e^{\beta(J+h)} & e^{-\beta J} \\ e^{-\beta J} & e^{\beta(J-h)} \end{pmatrix}$$

$$\langle 1 | P | -1 \rangle = P_{1,-1} = e^{-\beta J}$$

$$\langle 1 | P | 1 \rangle$$

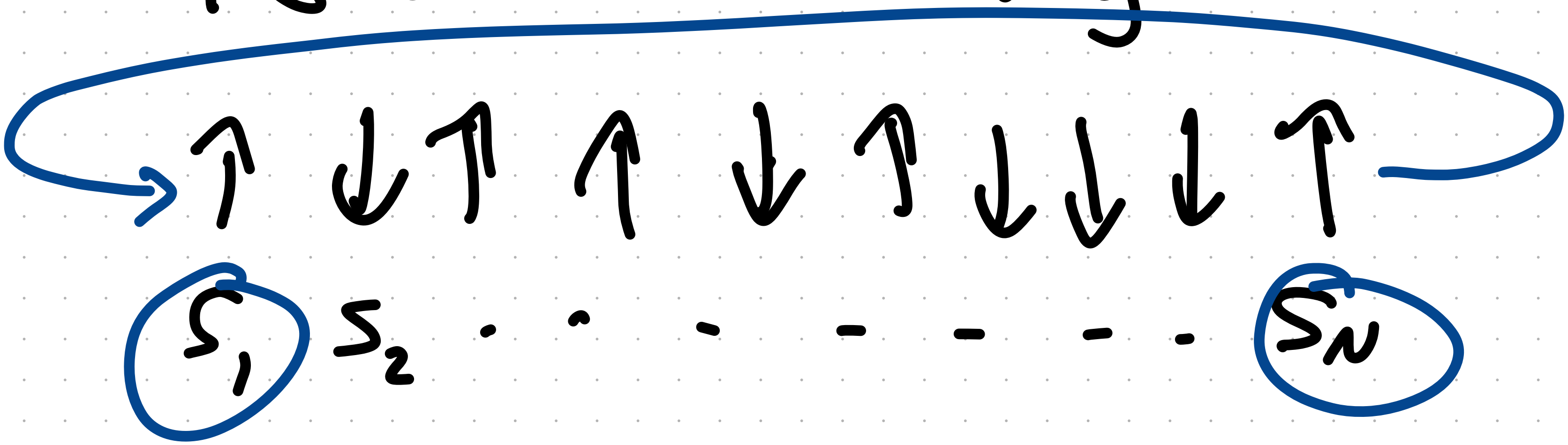
$$Z = \sum_{s_1, s_2, \ldots, s_N = \pm 1} \langle s_1 | P | s_2 \rangle \langle s_2 | P | s_3 \rangle \cdots \langle s_{N-1} | P | s_N \rangle$$

N^2 terms

$$\sum_{s_i} | s_i \rangle \langle s_i | = 1$$

$$Z = \sum_{s_1, s_N} \langle s_1 | P^{N-1} | s_N \rangle$$

Periodic boundary conditions



$$s_{N+1} = s_1$$

w/ PBC

$$Z = \sum_{s_1 \ldots s_N} \langle s_1 | P | s_2 \rangle \langle s_2 | P | s_3 \rangle \cdots \langle s_N | P | s_1 \rangle$$

$$= \sum_{s_1} \langle s_1 | \underline{P}^N | s_1 \rangle = \mathrm{Tr}\left(P^N\right)$$

$$\mathrm{Tr}(A) = \sum_{i=1}^{N} A_{ii}$$

$$\mathrm{Tr}(UXU^{-1}) = \mathrm{Tr}(U^{-1}UX) = \mathrm{Tr}(X)$$

$$\mathrm{Tr}(ABC) = \mathrm{Tr}(CAB) = \mathrm{Tr}(BCA)$$

$$P^N = P \cdot P \cdot P \cdots P$$

$$P = UDU^{-1} \qquad D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_d \end{pmatrix}$$

eigenvalues are on diagonal

$$P^N = (UDU^{-1})(UDU^{-1})\cdots(UDU^{-1}) = UD^NU^{-1}$$

$$\mathrm{Tr}(P^N) = \mathrm{Tr}(D^N) = \sum_{i=1}^{d} \lambda_i^N$$

for this case

$$Z = \lambda_1^N + \lambda_2^N \leftarrow \text{eigenvalues of } P$$

PBC

$$\mathrm{Det}[P - \lambda I] = 0$$

$$\begin{vmatrix} e^{\beta(J+h)} - \lambda & e^{-\beta J} \\ e^{-\beta J} & e^{\beta(J-h)} - \lambda \end{vmatrix} = 0$$

$$\lambda_{\pm} = e^{\beta J}\left[\cosh(\beta h) \pm \sqrt{\sinh^2(\beta h) + e^{-4\beta J}}\right]$$

$$\cosh(x) = \frac{e^{-x} + e^{+x}}{2} \qquad \sinh(x) = \frac{e^{+x} - e^{-x}}{2}$$

$$Z = \lambda_+^N + \lambda_-^N = \lambda_+^N\left[1 + \left(\frac{\lambda_-}{\lambda_+}\right)^N\right] \approx \lambda_+^N$$

$$f(h,\beta) = -\frac{k_B T}{N} \ln \lambda_+^N$$

$$= -k_B T \ln(\lambda_+)$$

$$m(h,\beta) = \frac{\partial \log z}{\partial(\beta h)} = \frac{\sinh(\beta h) + \frac{\sinh(\beta h)\cosh(\beta h)}{\sqrt{\sinh(\beta h)^2 + e^{-4\beta J}}}}{\cosh(\beta h) + \sqrt{\sinh(\beta h)^2 + e^{-4\beta J}}}$$

as $h \to 0$ $\quad \sinh(0) = 0$ so
$\quad \cosh(0) = 1$

$m(h \to 0) = 0$ if $\beta < \infty$, if $T = 0$

if $T \to 0$ $\quad e^{-4\beta J} \to 0$

$$m \to \frac{\sinh(\beta h) \pm \cosh(\beta h)}{\cosh(\beta h) \pm \sinh(\beta h)}$$

$$= \pm 1$$