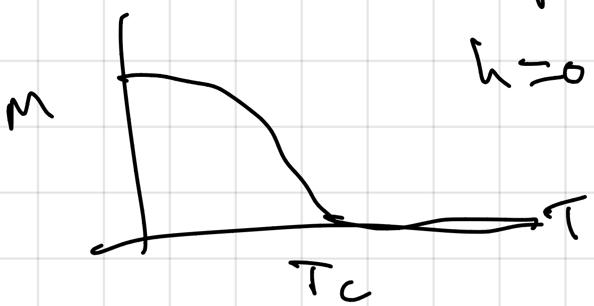
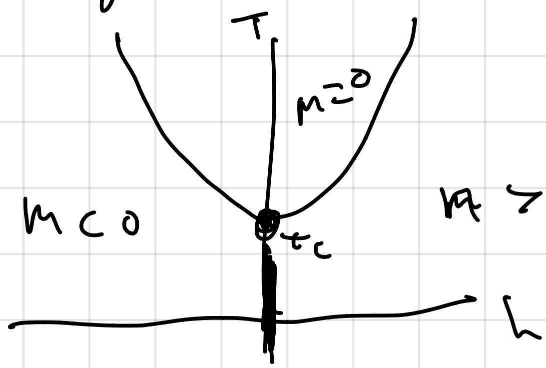
Phase Transitions Pt 3

Reminder: Want to know if Ising model has critical point, below which in Temp there is a spontaneous magnetization





Try to find in mean field, where spin sees average mag of neighbors: Result

$$H^{MF} = Jm^2 NZ - (h + 2mJZ)\sum_{i=1}^{N} s_i$$

$2Z =$ coord #

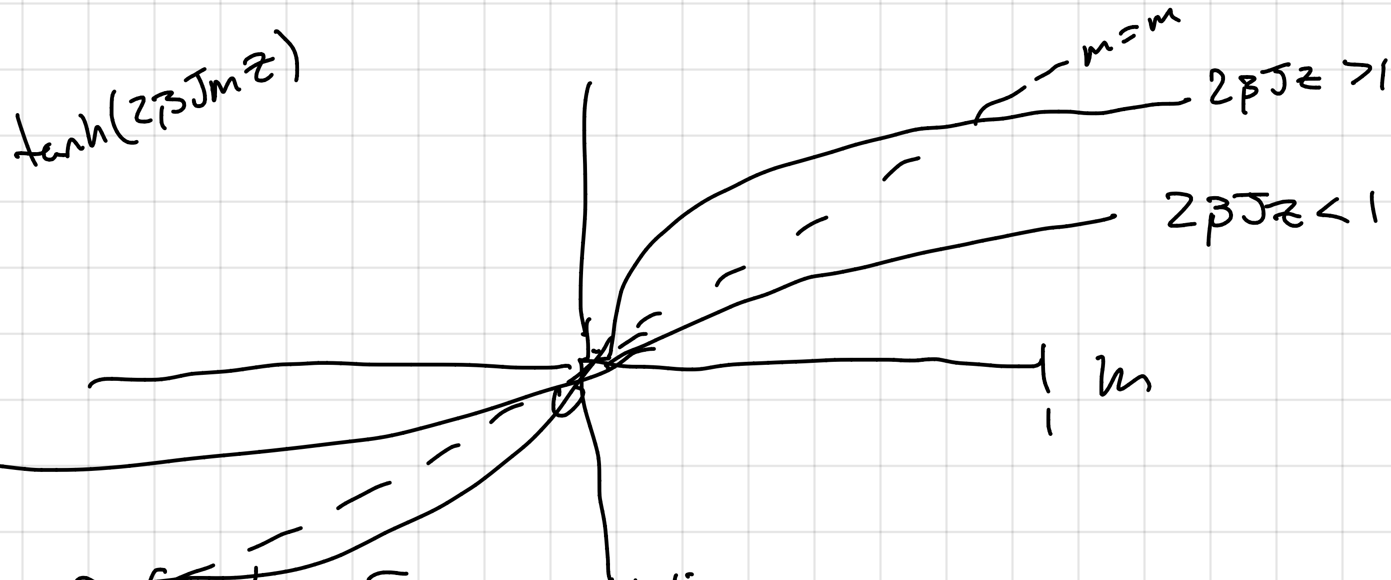
$$\text{So } Z = \sum_{s_1, s_2 \ldots s_N} e^{-\beta J m^2 N Z} \cdot e^{+\beta(h+2mJZ)\sum s_i}$$

$$= e^{-\beta J m^2 NZ} \left[2\cosh\left((h+2mJZ)\beta\right)\right]^N$$

$$\langle m \rangle = \frac{1}{N}\frac{\partial \log Z}{\partial \beta h} = \frac{\partial \log(\cosh(h+2mJZ)\beta)}{\partial \beta h} = \frac{\sinh(h+2mJZ)\beta}{\cosh(h+2mJZ)\beta}$$

$$m = \tanh(h+2mJZ)\beta$$

No analytical solution, can get numerically
@ $h=0$, spontaneous mag?



for low T, 3 solutions,
m=0, trivial not as interesting
other solution where lines cross

$2\beta J z = 1$ separates the regimes, & hence

$k_B T_C = 2Jz$ gives predicted critical point $\quad T_c = \frac{2Jz}{k_B}$

For 1d, wrong, no $T_c > 0$

for 2D, $T_c = 2.269\, J/k_B$ so
MF overestimate w/ $T_c = 4$, b/c neglects flucs
Mean field Better as $d \to \infty$, in general
[Ising, exact d=4 & above]

Lets expand $f(m,\beta)$ near $m=0$ near $\beta = \beta_c$.

$$f(m,\beta) \overset{MF}{\approx} -\frac{k_B T \log Z}{N} = Jm^2 z - \frac{1}{\beta}\log\left(2\cosh[(\cancel{h} + 2mzJ)\beta]\right)$$

$$2\cosh(x) = e^x + e^{-x} \approx 2\left(\sum \frac{1}{n!}(x^n + (-x)^n)\right) = 2 + x^2 + \cancel{2}\frac{x^4}{4!}\cdot 2$$

$$\log(2\cosh(x)) = \log 2 + \log(\cosh(x))$$

$$\log(1+x') = x' - \frac{x'^2}{2} + \frac{x'^3}{3} - \cdots, \quad x' = \frac{x^2}{2} + \frac{x^4}{4!} + \cdots$$

near m

So $f(0,\beta) \approx C_0 + C_2 m^2 + C_4 m^4 + \cdots$

$$C_2 = Jz \quad + \_\_\_\_$$

$$-\frac{1}{\beta}\log[2\cosh(2mJz\beta)] \approx -\frac{1}{\beta}\log\left[1 + \frac{1}{2}(2Jz\beta)^2 m^2\right] + C$$

$$\approx m\left[-\frac{4J^2}{2}z^2\beta + \text{const} + \cdots\right]$$

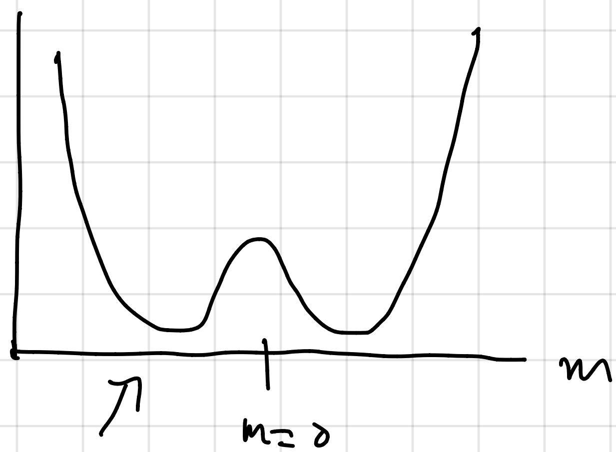
$$C_2 = Jz - 2J^2z^2\beta$$

neg when $2J^2z^2\beta > Jz$

or $2Jz\beta > 1$

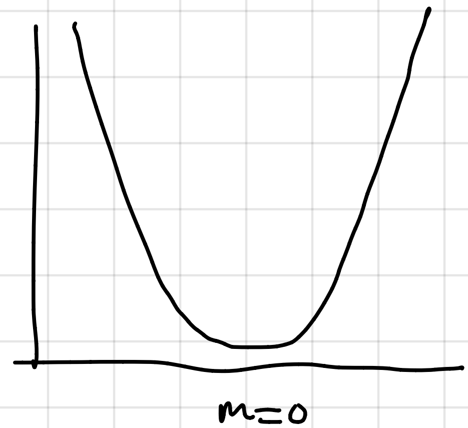
In this case

$f(\beta)$



two minima, $m \neq 0$

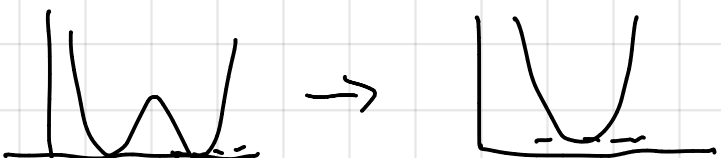
if $2J\beta z < 1$



In many cases, physicists will posit a free energy like this based on the symmetries of the problem, then predict critical behavior

For any problem where, near phase transition

$$f(T) \propto C_1 + C_2 m^2 + C_4 m^4$$



ask where $\left.\frac{df}{dm}\right|_{m_0} = 0$ & $m_0 > 0$

$$0 = 2 m_0 C_2 + 4 C_4 m_0^3 \Rightarrow$$

$$m_0^2 = \frac{-2c_2}{4c_4} \Rightarrow m_0 \sim \sqrt{\frac{-c_2}{2c_4}}$$

$\beta_c = \frac{1}{2Jz}$, $\quad c_2 = Jz - 2J^2 z^2 \beta$

$$= \frac{1}{2\beta_c} - \frac{\beta}{2\beta_c^2} = \frac{1}{2}\left(T_c - \frac{T_c^2}{T}\right)$$

$$= \frac{T_c}{2T}(T - T_c)$$

$$m_0 \sim (T - T_c)^{1/2}$$

critical exponent β

The behavior near a critical point is very strange, and involves many properties "diverging," meaning they become infinitely large as one got close to the transition

We can characterize the transition we are observing by its critical exponents, the power of the divergence, ie this β. What makes phase transitions fascinating is the Universality, Seemingly different problems had/have same critical exponents, in particular

$$C_V \sim |T - T_C|^{-\alpha}, \quad C_V = \left(\partial E / \partial T\right)$$

$$K_T \sim |T - T_C|^{-\gamma}, \quad K_T = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)$$

$$P - P_c \sim (\rho - \rho_c)^{\delta} \operatorname{sign}(\rho - \rho_c)$$

$$\rho_L - \rho_G \sim |T_c - T|^{\beta}$$

Table 16.1 compares ising model to gas, liquid
seemingly same exponents
$k_T \sim \chi = \partial m/\partial h$
$P \sim h = \partial A/\partial m$
$C_v \sim C_v$
(not shown, $\rho_L - \rho_G \sim m$)

Scientists started to notice this behavior, and in the mid 60's, theories started to emerge on the origin of these trends

Widom was able to derive connections between the exponents by a scaling theory, meaning looking at how the free energy changes w/ thermodynamic parameters

Eg $\quad 2-\alpha = 2\beta+\gamma$ [scaling relation]
[maybe next class]

this also follows from "renormalization group theory". Turns out $\alpha = 0$ [see pg 622]

MF $\alpha = 0$, $\beta = 1/2$, $\gamma = 1$, $\delta = 3$ [same as van der waals]

real (3d) $\alpha \approx .1$, $\beta = .34$, $\gamma = 1.35$, $\delta = 4.2$

These systems have same critical exponents because they are in the same "universality class" and have same types of symmetries:

The things that end up mattering are:

1) Dimension of order parameter (n)

magnetization / density = scalar, $n = 1$

2) Dimension in which the system lives:

ie 3d ising model / liquid gas, $d = 3$

Mean field models give result for $d \to \infty$

eg in MFT, $n = 1$,

we saw $\beta = 1/2$

Another example:

$$m = \tanh(\beta(2Jmz + h))$$

$$h = k_B T \tanh^{-1}(m) - 2mJz \quad \text{(like pressure)}$$

expanding w/ $\tanh^{-1} \sim x + x^3/3 + \cdots$ near $m = 0$

check

$$h \approx k_B T\left(m + m^3/3\right) - \lambda \bar{J}_z$$

$$= m k_B\left(T - 2\bar{J}_z/k\right) + k_B T/3\, m^3$$

$$= m k_B\left(T - T_c\right) + \frac{k_B T}{3} m^3$$

so $h \sim m^3$, $\delta = 3$

$$\chi = \partial m/\partial h = 1/\partial h/\partial m$$

$$\partial h/\partial m \approx k_B\left(T - T_c\right) + k_B T m^2$$

for $T > T_c$, $m = 0$, so as

$$\lim_{T \to T_c^+} \chi = 1/k_B\left(T - T_c\right) \sim \left(T - T_c\right)^{-1}$$

so $\gamma = 1$