

# Phase transitions, Pt 2

Last time: discussed general physics of phase transitions

We introduced the Ising Model as a simple way to study magnetization phase transition

$$H = -\frac{1}{2} \sum_{\langle i,j \rangle} J s_i s_j - h \sum_i s_i \quad (n \text{ dimensions})$$

$$\& \text{1d, w/PBC } H = -\sum_{i=1}^N (J s_i s_{i+1} + \frac{h}{2} (s_i + s_{i+1}))$$

Q: when is there a spontaneous magnetization transition:  $m = M/N = \langle \sum s_i \rangle / N = \langle s_i \rangle > 0, \text{ w } h=0$

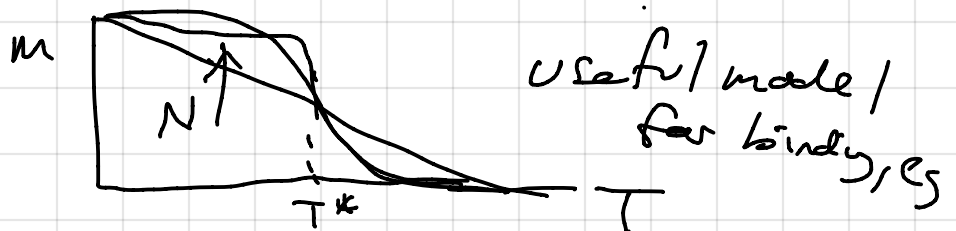
Said, should not exist for 1d ising model

For 1d model,  $h=0$ , said

$$Z \propto (2 \cosh(\beta J))^N$$

$$\text{and } f = F/N = -k_B T / N \log Z = -\frac{1}{\beta} \log(2 \cosh(\beta J))$$

Note: w/  $h > 0$



How can we "solve" for  $h=0$ .  $\Rightarrow$  compute  $Z$   
or derivatives of  $\log Z$

Need a new technique called transfer matrices

$P_{s,s'}$  is the matrix w/ entries  $e^{\beta J(s s') + \beta h(s+s')/2}$

$$\text{so } \langle 1|P|1\rangle = e^{\beta J + \beta h} \quad \langle -1|P|-1\rangle = e^{\beta J - \beta h}$$

$$\langle 1|P|-1\rangle = \langle -1|P|1\rangle = e^{-\beta J}$$

$$P = \begin{pmatrix} e^{\beta(J+h)} & e^{-\beta J} \\ e^{\beta J} & e^{\beta(J-h)} \end{pmatrix}$$

$$\text{so } Z = \sum_{s_1, \dots, s_N} \langle s_1|P|s_2\rangle \langle s_2|P|s_3\rangle \dots \langle s_N|P|s_1\rangle$$

Spin eigenvectors are complete i.e.  $\sum_{s_i} |s_i\rangle \langle s_i| = 1$

$$\Rightarrow Z = \sum_{s_1} \langle s_1|P^N|s_1\rangle = \text{Tr}(P^N)$$

$\text{Tr}(M^N) = \text{Tr}(U D^N U^{-1}) = \sum_{i=1}^N \lambda_i^N$ ,  $\lambda_i$   
are eigenvalues of  $M$  if  $M$  diagonalizable

$$\Rightarrow Z = \lambda_1^N + \lambda_2^N$$

$$\text{Det}[P - \lambda I] = 0$$

$$\lambda = e^{\beta J} \left( \cosh(\beta h) \pm \sqrt{\sinh^2(\beta h) + e^{-4\beta J}} \right)$$

$$\text{as } N \rightarrow \infty \quad Z \approx \lambda_+^N$$

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \quad \cosh(x) = \frac{e^x + e^{-x}}{2}$$

and  $f(h, \beta) = -\frac{1}{\beta} \log \left[ e^{\beta J} \left( \cosh(\beta h) + \sqrt{\sinh^2(\beta h) + e^{-4\beta J}} \right) \right]$

$$M(h, \beta) = \frac{\partial \log Z}{\partial \beta h} = \frac{\sinh(\beta h) + \frac{1}{2} (\sinh^2(\beta h) + e^{-4\beta J})^{-1/2} \cdot 2 \sinh(\beta h) \cosh(\beta h)}{\cosh(\beta h) + \sqrt{\sinh^2(\beta h) + e^{-4\beta J}}}$$

as  $h \rightarrow 0$ ,  $\sinh(\beta h) = 0$ ,  $\cosh(\beta h) \rightarrow 1$

So  $\lim_{h \rightarrow 0} (M(h, \beta)) = 0$

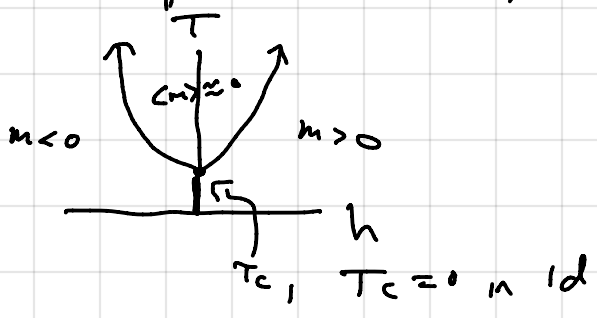
@  $\beta < \infty$ ,

if  $\beta \rightarrow \infty$  ( $T \rightarrow 0$ ),  $e^{-4\beta J} \rightarrow 0$  and

$$M \rightarrow \frac{\sinh(\beta h) \pm \cosh(\beta h)}{\cosh(\beta h) \pm \sinh(\beta h)} \rightarrow \pm 1$$

and which depends on  $h > 0$  or  $h < 0$

so phase transition / critical at  $T \rightarrow 0$ , then  $h \rightarrow 0$



What about higher dimensions? We can solve approx using a technique called "mean field theory, meaning each spin feels the avg effect of its neighbors  
lets call  $\delta s_i = s_i - m \Rightarrow s_i = \delta s_i + m$

$$\text{Then } H = -\frac{J}{2} \sum_{\langle ij \rangle} s_i s_j - h \sum_i s_i$$

$$= -\frac{J}{2} \sum_{\langle ij \rangle} (m + \delta s_i)(m + \delta s_j) - h \sum_i s_i$$

$m^2 + m(\delta s_i + \delta s_j) + \delta s_i \delta s_j$ 
 $\nearrow$  neglect

$$= -\frac{J}{2} \sum_{\langle ij \rangle} (m^2 + m(s_i - m) + m(s_j - m)) - h \sum_i s_i$$

(double counting)

$$= +J \sum_{\langle ij \rangle} (m^2) - \frac{mJ}{2} \sum_{\langle ij \rangle} (s_i + s_j) - h \sum_i s_i$$

$$= J m^2 N z - (h + 2mJz) \sum_{i=1}^N s_i$$

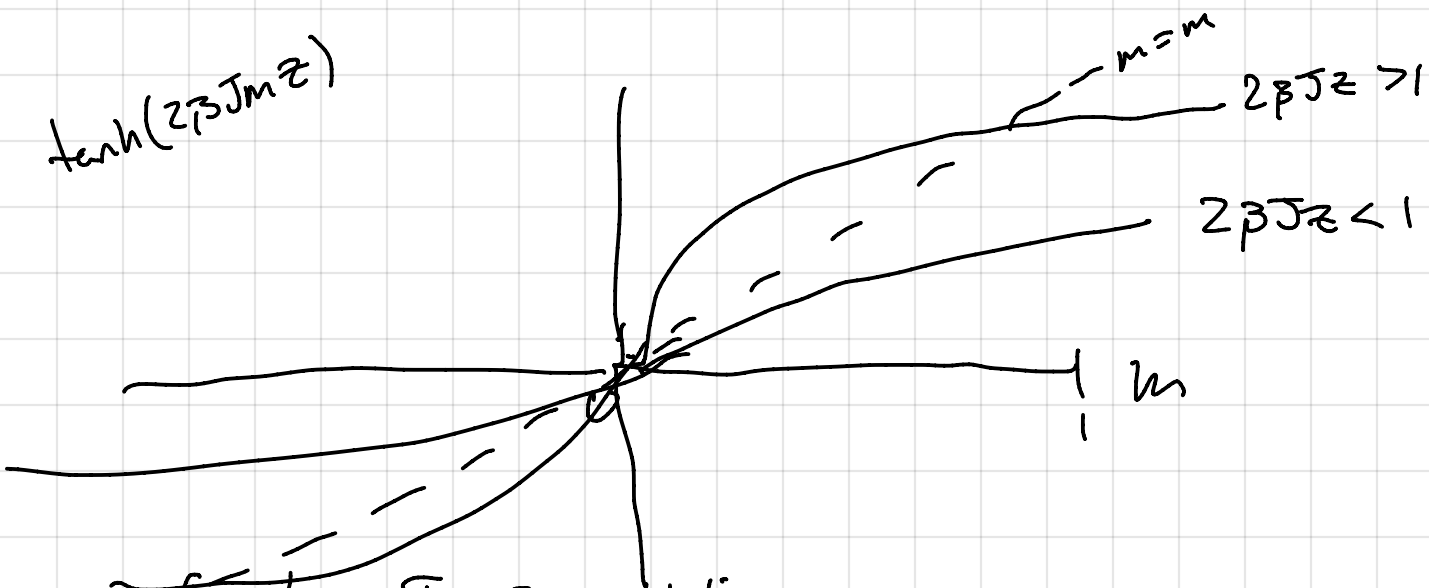
$$\text{So } z = \sum_{s_1, s_2, \dots, s_N} e^{-\beta J m^2 N z} \cdot e^{+\beta (h + 2mJz) \sum s_i}$$

$$= e^{-\beta J m^2 N z} [2 \cosh((h + 2mJz)\beta)]^N$$

$$\langle m \rangle = \frac{1}{N} \frac{\partial \log z}{\partial \beta h} = \frac{\partial \log(\cosh((h + 2mJz)\beta))}{\partial \beta h} = \frac{\sinh((h + 2mJz)\beta)}{\cosh((h + 2mJz)\beta)}$$

$$m = \tanh((h + 2mJz)\beta)$$

No analytical solution, can get numerically  
 @  $h=0$ , spontaneous mag?



for low  $T$ , 3 solutions,  
 $m=0$ , trivial not as interesting  
 other solution where lines cross

$2\beta J z = 1$  separates the regimes, & hence

$k_B T_c = 2Jz$  gives predicted  
 critical point

For 1d, wrong, no  $T_c > 0$