Phase transitions:

Part 4

Spatial correlations:

Important physical quantity in science, especially phase transitions are spatial correlations. This means if something is true at place $\vec{R}$, how likely is it to be true at place $\vec{R}+\vec{r}$. This was like $g(r)$, how much more likely are we to find a particle than any a given distance away from something

We also measured this as

$$\langle \rho(r)\, \rho(r')\rangle \quad (\text{or } \langle \delta\rho(r)\, \delta\rho(r')\rangle)$$

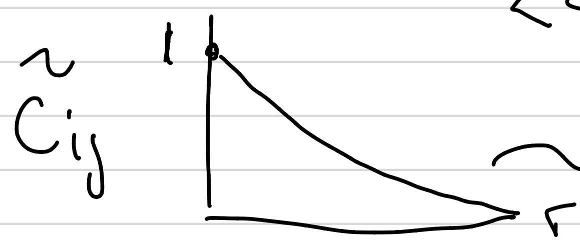
In a lattice model, we compute the following related quantity

$$C_{ij} = \langle (s_i - \langle s_i\rangle)(s_j - \langle s_j\rangle)\rangle = \langle s_i s_j\rangle - \langle s_i\rangle\langle s_j\rangle$$

called the spin-spin correlation function

Goes to zero when $\langle s_i s_j \rangle \to \langle s_i \rangle \langle s_j \rangle$,
which happens when uncorrelated,
eg $|j-i| \gg 0$, sometimes normalize

$$C_{ij}, \quad \tilde{C}_{ij} = \frac{\langle s_i s_j \rangle - \langle s_i \rangle \langle s_j \rangle}{\langle s_i^2 \rangle - \langle s_i \rangle^2} \quad \text{st}$$



$\sim e^{-r/\xi} \leftarrow$ length scale of correlations

so volume ξ^d correlated

spins correlated w/ spin 1, eg

$$= \sum_{j=2}^{N} C_{1j} \qquad \text{(could have chosen any spin)}$$

This is connected to the suceptibility

$$\chi = \frac{1}{N} \langle (\delta M)^2 \rangle \quad \text{where} \quad \delta M = \sum_{i=1}^{N} (s_i - \langle s_i \rangle)$$

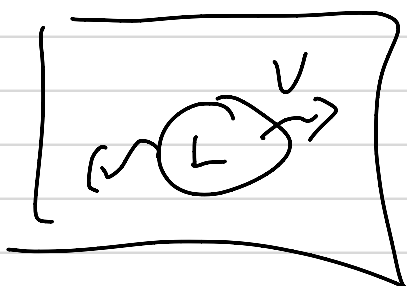
$$= \frac{1}{N} \sum_{i,j} \langle s_i s_j \rangle - \langle s_i \rangle \langle s_j \rangle$$

$$= \frac{N}{N} \cdot \sum_j \langle s_1 s_j \rangle - \langle s_1 \rangle \langle s_j \rangle = \sum_j C_{1j}$$

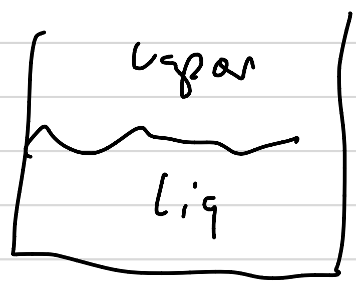
But we already saw that χ diverges @ a phase transition, so this is a signature of "long range correlations"

2 ways this can diverge

1) 2 phase coexistence



water droplet



as these get more sharp w/ increasing N, variance gets maximized

Near critical point, no distinction between two phases, so the divergence becomes large means correlations become long range usually $G(r_{ij}) \equiv c_{ij} \sim \dfrac{e^{-r/\xi}}{r^{d-2+\eta}}$ fit to

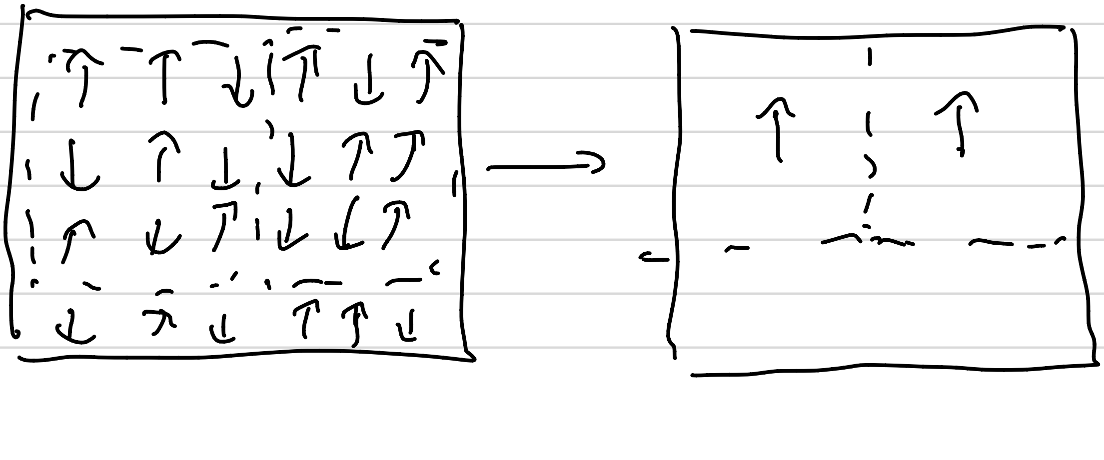
and as critical point is approached
$\xi \sim |T - T_c|^{-\nu}$

The fact that this length scale gets big means the system looks the same on small & large length scales, which leads to the renormalization group idea!

Renormalization Group

"coarse grain" over section of system
try to have partition function look the same

Eg



(zero field

then eg $H = \sum_{\langle ij \rangle} J s_i s_j \rightarrow \sum_{\langle i', j' \rangle} J' s_{i'} s_{j'}$

Then you can repeat this procedure & iterate
If $J \rightarrow J'$ process converges, then
this is at a "fixed point" and it
corresponds to a phase transition.
From the equations that generate $J \rightarrow J'$,
we can get the critical exponents too!

We'll illustrate in 1d ising model (no
spontaneous mag transition) & discuss the
result in 2d [1 of different possible procedures]

@ zero field

$$Q(K, N) = \sum_{s_1, \ldots s_N = \pm 1} \exp[\overset{K = \beta J}{K}(s_1 s_2 + s_2 s_3 + \cdots + s_N s_1)]$$

In mean field, averaged over all other

degrees of freedom but 1. Here, remove finite degrees of freedom.

Eg: Sum over even spins

$\bar{1}\ \bar{2}\ \bar{3}\ \bar{4}\ \bar{5}\ \bar{6} \rightarrow \bar{1}\ \bar{3}\ \bar{5}$

$$Q(K,N) = \sum_{s_1, s_3 \ldots} \left(\exp[K(s_1+s_3)] + \exp[-K(s_1+s_3)]\right)$$
$$\times \left(\exp[K(s_3+s_5)] + \exp[-K(s_3+s_5)]\right)$$
$$\times \ldots$$

want to express in same as original form, if possible, but w/ new K' (new inverse temperature, or J)

Would be true if $e^{K(s+s')} + e^{-K(s+s')} = f(K) e^{K' s s'}$

for all $s s'$

then $Q(K,N) = f(K)^{N/2}\, Q(K', N/2)$

[Kadanoff transformation]

finding this is possible in 1d

[in higher d, have to approximate]

if s, s' in same direction

$$e^{2k} + e^{-2k} = f(k)\, e^{k'} \qquad ①$$

if opposite

$$2 = f(k)\, e^{-k'} \Rightarrow f(k) = 2e^{k'} \qquad ②$$

can solve for k' & $f(k)$

② → ① $\quad 2e^{2k'} = e^{2k} + e^{-2k}$

$$\Rightarrow k' = \frac{1}{2}\log(\cosh(2k))$$

plugging back in to 2

$$f(k) = 2\cosh^{1/2}(2k)$$

Consider $\ln Q$ (a sort of free energy)

expect to grow $\propto N$. define $g(k) = \frac{1}{N}\ln Q$

[intensive free energy]

$$g(k) = \frac{1}{N}\cdot\Big[\frac{N}{2}\ln f(k) + \underbrace{\ln Q(k', N/2)}_{N/2\, g(k')}\Big]$$

$\Rightarrow$ recursion:

$$g(k') = 2g(k) - \ln f(k)$$
$$= 2g(k) - \ln\left(2\sqrt{\cosh(2k)}\right)$$

If we know $Q(k)$ for 1 value of k, we can find it for other values

In this renormalization, k' is always less than k

$\{0.5 \ln \cosh(2k) = k' < k\}$

Alternatively, swap k & k'

$$k = \tfrac{1}{2}\cosh^{-1}(\exp(2k'))$$

& solve for $g(k)$

$$g(k) = \tfrac{1}{2} g(k') + \tfrac{1}{2}\ln 2 + \tfrac{1}{2}\ln\left(\cosh(\cosh^{-1}(\exp(2k'))\right)$$
$$= \tfrac{1}{2} g(k') + \tfrac{1}{2}\ln 2 + k'/2$$

What do we do with this? For small k', eg $k' = 0.01$, high temp or low J means spins close to uncorrelated $Q(k', N) \approx 2^N$, $g(k') = \ln 2$
≈ 0.693

$$\Rightarrow k = 0.100334$$
$$g(k) = 0.698147$$

keep iterating & k keeps growing

during the iteration, $g(k)$ gets closer & closer to the exact $g(k)$ for that value of k [in 1d, can compute exactly

For no field & large N, we'd seen

$$Q(\beta, N) = (2\cosh(\beta J))^N$$
$$= (e^k + e^{-k})^N$$
$$\ln Q/N = \ln(e^k + e^{-k}) = k + \ln(1+e^{-2k})]$$

for large k, $g(k) \approx k$

can start w/ eg $g(10) \approx 10$ & iterate other equations to get smaller k results

Errors apparently grow in this procedure, but the picture is

$$\underset{k=0}{x} \longrightarrow \longrightarrow \longrightarrow \longrightarrow \underset{k=\infty}{x}$$

renormalization flow

$$\text{or } \underset{k=\infty}{x} \longrightarrow \longrightarrow \longrightarrow \longrightarrow \longrightarrow \underset{k=0}{x}$$

for $k=0$ or $k=\infty$, params don't change "fixed point"

In 2d

$$\overset{\times}{k=0} \leftarrow \leftarrow \leftarrow \leftarrow \overset{\times}{k_c} \rightarrow \rightarrow \rightarrow \rightarrow \overset{\times}{k=\infty}$$

unstable fixed point at
$k_c = 0.50698$ (for a similar procedure see Chandler Ch 5)

where exact value $J/k_B T_c = k_c = 0.44069$

See book pg 261 that shows # iterations needed as get closer & closer to k_c grows, connecting to a growing length scale as T_c approached