Phase transitions:

Part 2

Last time: discussed general physics of phase transitions

We introduced the Ising Model as a simple way to study magnetization phase transition

$$H = -\tfrac{1}{2}\sum_{\langle i,j\rangle} J s_i s_j - h\sum_i s_i \qquad \text{(n dimensions)}$$

& 1d, w/ PBC $\quad H = -\sum_{i=1}^{N}\left(J s_i s_{i+1} + \frac{h}{2}(s_i + s_{i+1})\right)$

Q: when is there a spontaneous magnetization transition: $m = M/N = \langle \sum s_i \rangle / N = \langle s_i \rangle > 0$, w $h = 0$

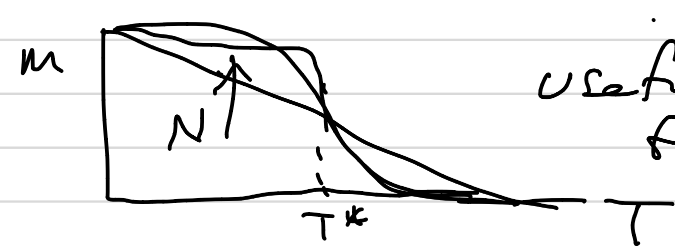
Said, should not exist for 1d ising model

For 1d model, $h=0$, said

$$Z \propto (2\cosh(\beta J))^N$$

and $f = F/N = -k_B T/N \log Z = -\frac{1}{\beta}\log(2\cosh(\beta J))$

Note: w/ $h > 0$



useful model for bindings, es

But what about $h \neq 0$?

Need a new technique called transfer matrices

$P_{s,s'}$ is the matrix w/ entries $e^{\beta J(ss') + \beta h(s+s')/2}$

so $\langle 1|P|1\rangle = e^{\beta J + \beta h}$ $\quad \langle -1|P|-1\rangle = e^{+\beta J - \beta h}$

$\langle 1|P|0\rangle = \langle 0|P|1\rangle = e^{-\beta J}$

$$P = \begin{pmatrix} e^{\beta(J+h)} & e^{-\beta J} \\ e^{-\beta J} & e^{\beta(J-h)} \end{pmatrix}$$

so $Z = \sum_{s_1,\dots,s_N} \langle s_1|J|s_2\rangle\langle s_2|J|s_3\rangle \cdots \langle s_N|J|s_1\rangle$

Spin eigenvectors are complete ie $\sum_{s_i} |s_i\rangle\langle s_i| = 1$

$\Rightarrow Z = \sum_{s_1} \langle s_1|P^N|s_1\rangle = \mathrm{Tr}(P^N)$

$\mathrm{Tr}(M^N) = \mathrm{Tr}(UD^N U^{-1}) = \sum_{i=1}^{N} \lambda_i^N$, λ_i are eigenvalues of M if m diagonalizable

$\Rightarrow Z = \lambda_1^N + \lambda_2^N$ $\quad \mathrm{Det}[P - \lambda I] = 0$

$\lambda = e^{\beta J}\left(\cos(\beta h) \pm \sqrt{\sinh^2(\beta h) + e^{-4\beta J}}\right)$

as $N \to \infty$ $\quad Z \approx \lambda_+^N$

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \qquad \cosh(x) = \frac{e^x + e^{-x}}{2}$$

and $$f(h,\beta) = -\frac{1}{\beta}\log\left[e^{\beta J}\left(\cosh(\beta h) + \sqrt{\sinh^2(\beta h) + e^{-4\beta J}}\right)\right]$$

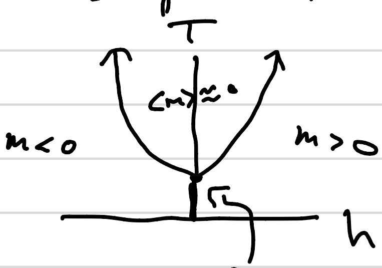
$$m(h,\beta) = \frac{\partial \log z}{\partial \beta h} = \frac{\sinh(\beta h) + \frac{1}{2}\left(\sinh^2(\beta h) + e^{-4\beta J}\right)^{-1/2}\cdot 2\sinh\beta h\cosh\beta h}{\cosh(\beta h) + \sqrt{\sinh^2(\beta h) + e^{-4\beta J}}}$$

as $h \to 0$, $\sinh(\beta h) = 0$, $\cosh(\beta h) \to 1$

So $\lim_{h\to 0}(m(h,\beta)) = 0$
@ $\beta < \infty$,
if $\beta \to \infty$ ($T \to 0$), $e^{-4\beta J} \to 0$ and

$$m \to \frac{\sinh(\beta h) \pm \cosh(\beta h)}{\cosh(\beta h) \pm \sin(\beta h)} \to \pm 1$$

and which depends on $h > 0$ or $h < 0$
so phase transition / critical point at $T \to 0$, then $h \to 0$



T_c, $T_c = 0$ in 1d

What about higher dimensions? We can solve approx using a technique called "mean field theory, meaning each spin feels the avg effect of its neighbors

lets call $\delta s_i = s_i - m \Rightarrow s_i = \delta s_i + m$

Then $H = -\frac{J}{2} \sum_{\langle ij \rangle} s_i s_j - h \sum_i s_i$

$$= -\frac{J}{2} \sum_{\langle ij \rangle} (m + \delta s_i)(m + \delta s_j) - h \sum_i s_i$$

$$\shortparallel$$

$$m^2 + m(\delta s_i + \delta s_j) + \cancel{\delta s_i \delta s_j} \xrightarrow{} 0 \text{ neglect}$$

$$= -\frac{J}{2} \sum_{\langle ij \rangle} \left(m^2 + m(s_i - m) + m(s_j - m)\right) - h \sum_i s_i$$

$$= +\frac{J}{2} \sum_{\langle ij \rangle} (m^2) - \frac{mJ}{2} \sum_{\langle ij \rangle} (s_i + s_j) - h \sum_i s_i$$

$$= J m^2 N z - (h + 2mJz) \sum_{i=1} s_i$$

So $Z = \sum_{s_1, s_2 \ldots s_N} e^{-\beta J m^2 N z} \cdot e^{+\beta(h + 2mJz)\sum s_i}$

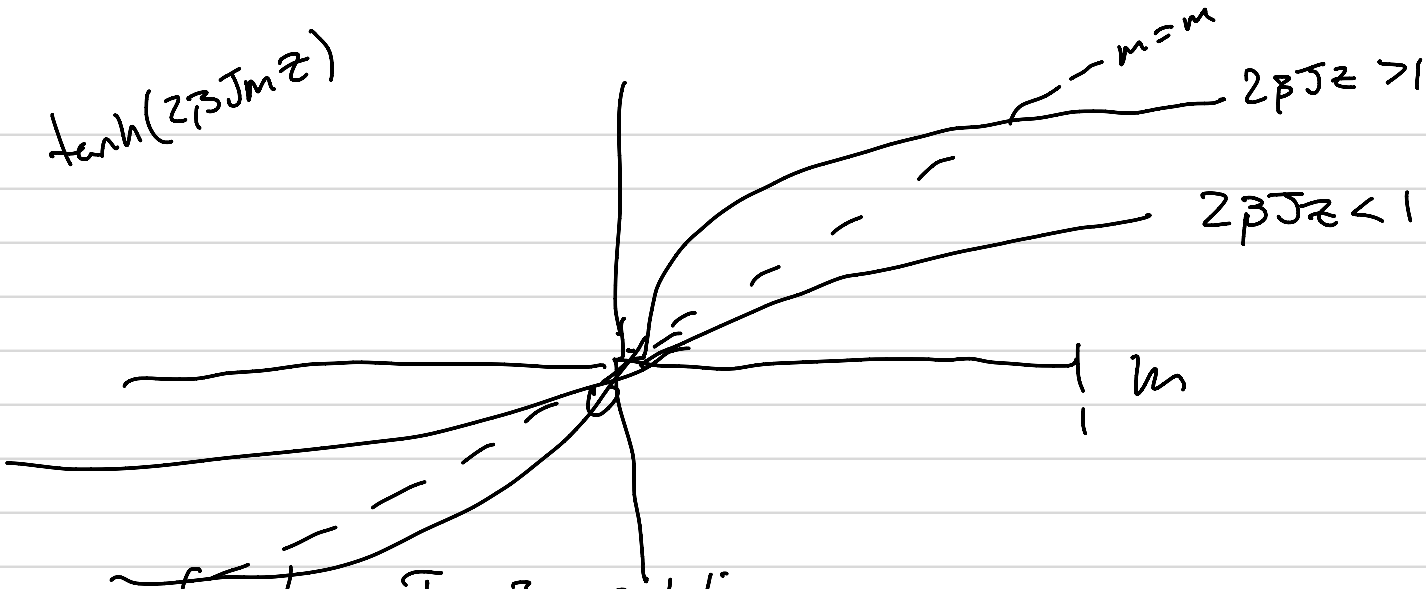
$$= e^{-\beta J m^2 N z} \left[2\cosh\left((h + 2mJz)\beta\right)\right]^N$$

$$\langle m \rangle = \frac{1}{N} \frac{\partial \log Z}{\partial \beta h} = \frac{\partial \log(\cosh(h + 2mJz)\beta)}{\partial \beta h} = \frac{\sinh(h + 2mJz)\beta}{\cosh(h + 2mJz)\beta}$$

$$m = \tanh(h + 2mJz)\beta$$

No analytical solution, can get numerically

@ $h = 0$, spontaneous mag?



for low T, 3 solutions,
$m=0$, trivial not as interesting
other solution where lines cross

$2\beta Jz = 1$ separates the regimes, & hence

$k_B T_c = 2Jz$ gives predicted critical point

For 1d, wrong, no $T_c > 0$