HW 4

1a: $\Omega_L^{start} = \binom{N_{Left}}{N_{Red}} = \binom{N_{Left}}{N_{Left}} = 1$

$\Omega_R^{start} = \binom{N_{right}}{N_{white}} = \binom{N_{Right}}{N_{Right}} = 1$

$\Omega^{start} = \Omega_L^{start} \cdot \Omega_R^{start} = 1$

$S_{start} = k_B \ln \Omega^{start} = 0$

$\Omega_{final} = \binom{N_{Left} + N_{right}}{N_{Red}} = \binom{N}{N_{Red}}$

$= \frac{N!}{N_{Red}!(N - N_{Red})!} = \frac{N!}{N_{Red}!\, N_{white}!} = \binom{N}{N_w}$

$\Delta S = S_{final} - 0 = k_B \ln \left[\frac{N!}{N_{Red}!(N - N_{Red})!} \right]$

Note: for final, once red or white known, other known

b) $\Omega_{start} = \binom{N_i}{N}$

$\Omega_{final} = \binom{N_f}{N}$

$$\Delta S = k_B \ln\left[\binom{N_f}{N}\right] - k_B \ln\left[\binom{N_i}{N}\right]$$

$$= k_B\Big[\ln N_f! - \ln[\cancel{N!}] - \ln[(N_f-N)!]$$

$$- \ln N_i! + \ln \cancel{N!} + \ln[(N_i-N)!]\Big]$$

$$\approx k_B\Big[N_f \ln N_f - N_f - N_i \ln N_i + N_i$$

$$-(N_f-N)\ln(N_f-N) + (N_f-N)$$

$$+(N_i-N)\ln(N_i-N) - (N_i-N)\Big]$$

$$= k_B\Big[N_f \ln N_f - N_i \ln N_i - (N_f-N)\ln[N_f-N)$$

$$+ (N_i-N)\ln[N_i-N]\Big]$$

$$= k_B\left[\ln \frac{N_f}{N_f-N} - N_i \ln \frac{N_i}{N_i-N} + N \ln\left(\frac{N_f-N}{N_i-N}\right)\right]$$

if $N_f >> N$ and $N_i >> N$

$$\approx k_B\left[0 - 0 + N \ln \frac{N_f}{N_i}\right]$$

if each box has volum v

an $V_f = N_f \cdot v$

$V_i = N_i v$

$$= N k_B \ln\left(\frac{N_f}{N_i} \cdot \frac{v}{v}\right) = NR \ln\left(\frac{V_f}{V_i}\right) \checkmark$$

2) a. i. $L = (N_d - N_u) \cdot l$

$N_d + N_u = N$

$\Rightarrow L = (N - 2N_u) l$

$L = (2N_d - N) l$

ii. $\mathcal{E} = -mgL$

$= -mgl[N - 2N_u]$

$= -mgl[2N_d - N]$

$\Rightarrow N_u = \frac{1}{2}\left[\frac{\mathcal{E}}{mgl} + N\right]$

$N_d = \frac{1}{2}\left[N - \frac{\mathcal{E}}{mgl}\right]$

iii) $$\Omega = \binom{N}{N_d} = \frac{N!}{N_d!\,N_u!}$$

$$= \frac{N!}{\left[\frac{1}{2}\left(\frac{\varepsilon}{mgl}+N\right)\right]!\left[\frac{1}{2}\left(N-\frac{\varepsilon}{mgl}\right)\right]!}$$

iv) $S = k_B \ln \Omega$

$$\frac{S}{k_B} \approx N\ln N - N - \left[\frac{1}{2}\left(\frac{\varepsilon}{mgl}+N\right)\right]\ln\left[\frac{1}{2}\left(\frac{\varepsilon}{mgl}+N\right)\right] + \frac{1}{2}\left(\frac{\varepsilon}{mgl}+N\right)$$
$$- \left[\frac{1}{2}\left(N-\frac{\varepsilon}{mgl}\right)\right]\ln\left[\frac{1}{2}\left(N-\frac{\varepsilon}{mgl}\right)\right] + \frac{1}{2}\left(N-\frac{\varepsilon}{mgl}\right)$$

$$= N\ln N - N - \left[\frac{1}{2}\left(\frac{\varepsilon}{mgl}+N\right)\right]\ln\left[\frac{1}{2}\left(\frac{\varepsilon}{mgl}+N\right)\right]$$
$$- \left[\frac{1}{2}\left(N-\frac{\varepsilon}{mgl}\right)\right]\ln\left(\frac{1}{2}\left(N-\frac{\varepsilon}{mgl}\right)\right)$$

$$\frac{S}{k_B} \simeq N\ln N - N - \left[\frac{1}{2}\left(\frac{\varepsilon}{mgl}+N\right)\ln\left[\frac{1}{2}\left(\frac{\varepsilon}{mgl}+N\right)\right]\right.$$

$$\left. - \left[\frac{1}{2}\left(N-\frac{\varepsilon}{mgl}\right)\ln\left(\frac{1}{2}\left(N-\frac{\varepsilon}{mgl}\right)\right)\right]\right.$$

$$\frac{1}{k_BT} = \left(\frac{\partial S/k_B}{\partial \varepsilon}\right) = -\cancel{\frac{1}{2}\left(\frac{\varepsilon}{mgl}+N\right)}\cdot\frac{1}{\cancel{\frac{1}{2}\left(\frac{\varepsilon}{mgl}+N\right)}}\cdot\frac{1}{2mgl}$$

$$-\frac{1}{2mgl}\cdot\ln\left[\frac{1}{2}\left(\frac{\varepsilon}{mgl}+N\right)\right]$$

$$-\cancel{\frac{1}{2}\left(N-\frac{\varepsilon}{mgl}\right)}\cdot\frac{1}{\cancel{\frac{1}{2}\left(N-\frac{\varepsilon}{mgl}\right)}}\cdot\frac{-1}{2mgl}$$

$$+\frac{1}{2mgl}\ln\left(\frac{1}{2}\left(N-\frac{\varepsilon}{mgl}\right)\right)$$

$$= \frac{1}{2mgl}\ln\left[\frac{N-\frac{\varepsilon}{mgl}}{N+\varepsilon/mgl}\right] = \frac{1}{2mgl}\ln\left[\frac{Nmgl-\varepsilon}{Nmgl+\varepsilon}\right]$$

vi) $$\frac{1}{k_B T} = \frac{1}{2mgl} \ln\left[\frac{Nmgl - \varepsilon}{Nmgl + \varepsilon}\right]$$

$$e^{\frac{2mgl}{k_B T}} = \frac{Nmgl - \varepsilon}{Nmgl + \varepsilon}$$

$$\Rightarrow (Nmgl + \varepsilon)e^{2\beta mgl} = (Nmgl - \varepsilon)$$

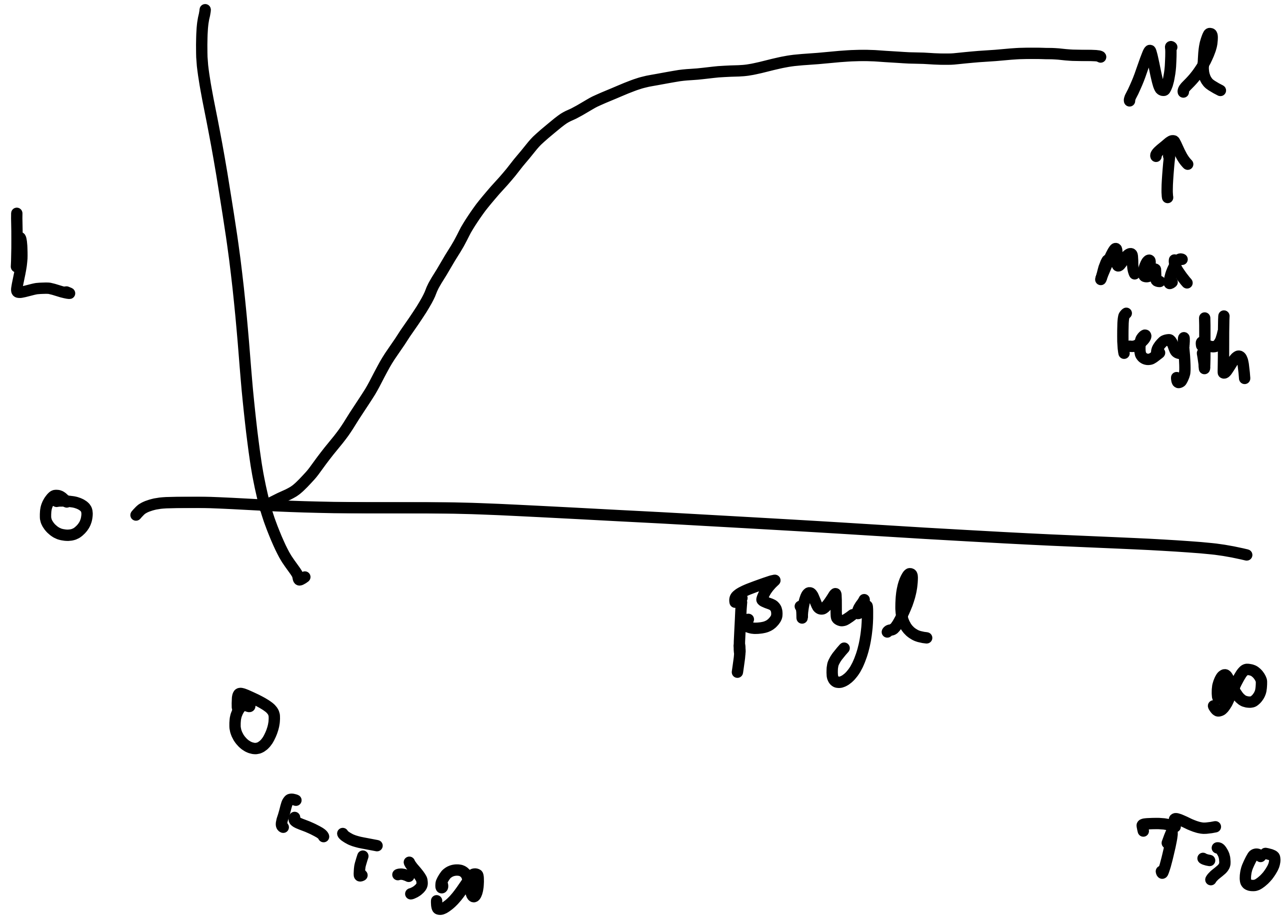
$$Nmgl\left[e^{2\beta mgl} - 1\right] = \varepsilon\left[-1 - e^{2\beta mgl}\right]$$

$$\varepsilon = -Nmgl\left[\frac{e^{2\beta mgl} - 1}{1 + e^{2\beta mgl}}\right]$$

$$= -Nmgl \tanh(\beta mgl)$$

$$L = \frac{\varepsilon}{-mg} = Nl \tanh(\beta mgl) \checkmark$$

vii)



b) i)

$$Z = \sum_{\varepsilon} \Omega(\varepsilon) e^{-\beta \varepsilon_i}$$

But ε can be written to depend only on N_d so that uniquely determines the state

$$Z = \sum_{N_d=0}^{\infty} \binom{N}{N_d} e^{+\beta mgl[2N_d - N]}$$

$$= e^{-\beta mglN} \cdot \sum_{N_d=0}^{\infty} \binom{N}{N_d} \left[e^{(2\beta mgl)}\right]^{N_d}$$

ii)

$$= e^{-\beta mglN} \left(1 + e^{2\beta mgl}\right)^N$$

$$= \left(e^{-\beta mgl} + e^{\beta mgl}\right)^N = \left[2\cosh(\beta mgl)\right]^N$$

iii) $\mathcal{E} = -\frac{\partial \ln z}{\partial \beta} = -\frac{1}{z}\frac{\partial z}{\partial \beta}$

$$= -\frac{N}{2\cosh(\beta mgl)} \cdot 2mgl \sinh(\beta mgl)$$

$$= -Nmgl \tan(\beta mgl)$$

iv)

$L = Nl \tanh(\beta mgl)$

v) $\mathrm{Var}\,\mathcal{E} = kT^2 \frac{\partial \mathcal{E}}{\partial T}$

$$\frac{\partial \mathcal{E}}{\partial T} = \frac{\partial \mathcal{E}}{\partial \beta}\frac{\partial \beta}{\partial T}$$

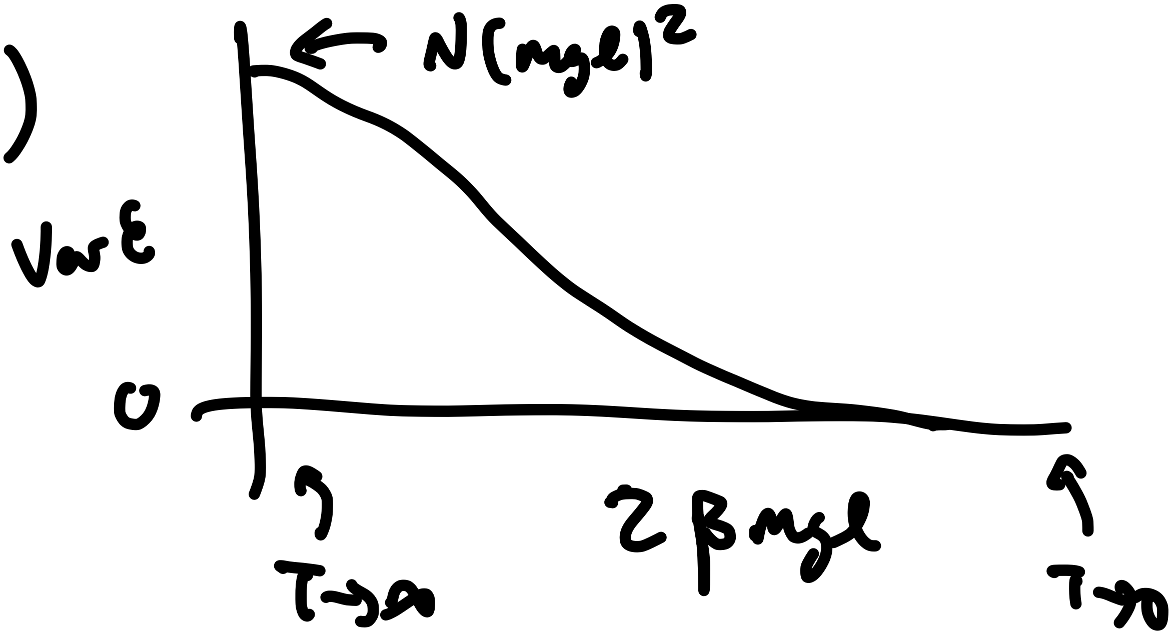
$$= -\frac{1}{T^2}\frac{\partial \mathcal{E}}{\partial \beta}$$

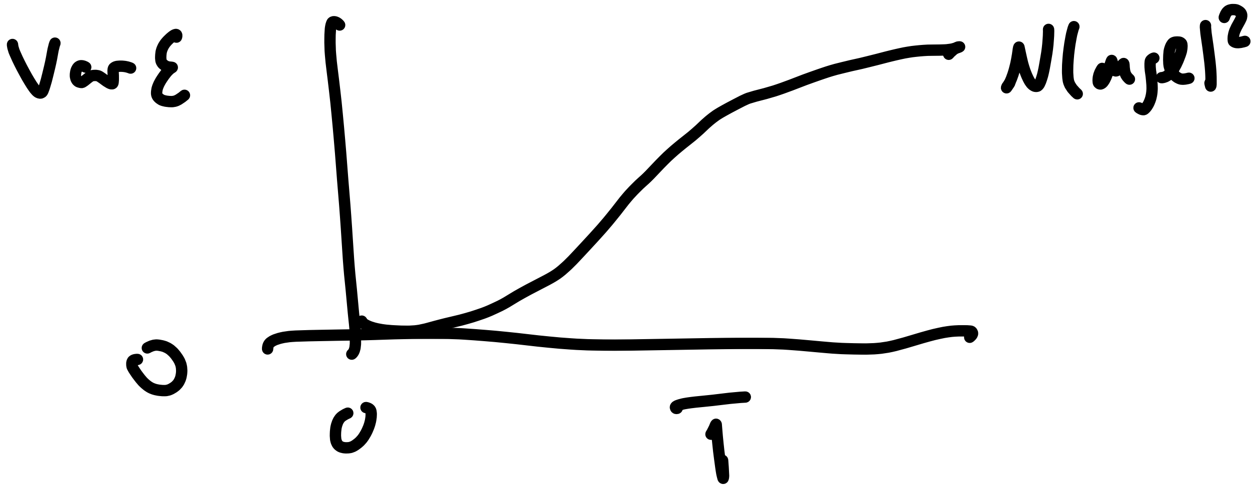
$$= -k_B \frac{\partial \mathcal{E}}{\partial \beta}$$

$$= Nmgl \cdot mgl \cdot \mathrm{sech}(2\beta mgl)$$

$$= N(mgl)^2 \mathrm{sech}(2\beta mgl)$$

vi)





Also note Var E has units of E^2

And $\sigma_E = mgl\sqrt{N}$ so,

$$\frac{|\sigma_E|}{|E|} \simeq \frac{mgl\sqrt{N}}{Nmgl\tanh(\beta mgl)} \sim \frac{1}{\sqrt{N}}$$ as we discussed in general