HW2 - Glen Hocky

① Gamma function

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x}\, dx$$

a) $\Gamma(1) = \int_0^\infty x^0 e^{-x} dx = \int_0^\infty e^{-x} dx$

$= -e^{-x} \Big|_0^\infty = 0 - (-1) = 1$ ✓

b) $\Gamma(z+1) = \int_0^\infty x^z e^{-x}\, dx$ $\left[\int u\,dv = uv - \int v\,du \right.$

$u = x^z \quad du = z x^{z-1}$

$dv = e^{-x} \quad \Rightarrow v = -e^{-x}$

$= \underbrace{-x^z e^{-x} \Big|_0^\infty}_{0-0} + \int_0^\infty e^{-x} \cdot z x^{z-1} dx = z\,\Gamma(z)$ ✓

c) $\Gamma(N+1) = N\,\Gamma(N) = N \cdot (N-1)\,\Gamma(N-1) = \cdots$

$= N(N-1)(N-2) \cdots \Gamma(1)$

$= N(N-1)(N-2) \cdots (1) = N!$ ✓

2 a) for $d=2$,

$$V_2 = \pi, \quad S_1 = 2\pi$$

$$S_1/V_2 = 2 = d \checkmark$$

$$V_3 = 4/3\pi \quad S_2 = 4\pi$$

$$S_2/V_3 = 3 = d \checkmark$$

b) i) $$I = \int_{-\infty}^{\infty} dx\, e^{-x^2}$$

$$I^d = \prod_{i=1}^{d} \int_{-\infty}^{\infty} dx_i e^{-x_i^2}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cdots dx_1 dx_2 \ldots e^{-\overbrace{\sum_{i=1}^{d} x_i^2}^{r^2}}$$

$$I^d = \int_{-\infty}^{\infty} dx^d e^{-r^2} = S_{d-1} \int_0^{\infty} r^{d-1} \underbrace{e^{-r^2}}_{f(r)} dr$$

ii) $$\int_0^{\infty} r^{d-1} e^{-r^2} dr = \int_0^{\infty} dy\, r^{d-2} e^{-y} = \int_0^{\infty} dy\, y^{\frac{d-2}{2}} e^{-y}/2$$

$y = r^2 \quad dy = 2r\,dr \Rightarrow dr = \frac{dy}{2r}$

$$= \frac{1}{2}\int_0^{\infty} dy\, e^{-y} y^{\frac{d}{2}-1} = \frac{1}{2}\Gamma(d/2)$$

iii) $I = \sqrt{\pi}$ so $I^d = \pi^{d/2}$

$$\Rightarrow \pi^{d/2} = \frac{1}{2}\Gamma(d/2) \cdot S_{d-1}$$

$$\Rightarrow S_{d-1} = \frac{2\pi^{d/2}}{\Gamma(d/2)}$$

iv) $V_d = S_{d-1}/d$

$$\Rightarrow V_d = \frac{2\pi^{d/2}}{d\,\Gamma(d/2)} = \frac{\pi^{d/2}}{(d/2)\Gamma(d/2)} = \underline{\frac{\pi^{d/2}}{\Gamma(d/2+1)}}$$

c) i) $\Gamma(t) = \int_0^\infty dx\, e^{-x} x^{t-1}$

$$\Gamma(1/2) = \int_0^\infty dx\, e^{-x} x^{-1/2} = \int_0^\infty 2r\,dr\, e^{-r^2} r^{-1} =$$

$\cancel{x} = r^2$, $dx = 2r\,dr$

$$= 2\int_0^\infty dr\, e^{-r^2} = \int_{-\infty}^{\infty} dr\, e^{-r^2} = \underline{\sqrt{\pi}}$$

ii) $V_3 = \pi^{3/2} / \Gamma(3/2+1)$

$$\Gamma(3/2+1) = 3/2\,\Gamma(1/2+1) = 3/2 \cdot \tfrac{1}{2}\,\Gamma(1/2) = \tfrac{3}{4}\sqrt{\pi}$$

$\Rightarrow V_3 = 4/3\,\pi$ ✓

remember
$\Gamma(N+1) = N!$

3) Microcanonical ideal gas

$$\Omega(N,V,\mathcal{E}) = \frac{\mathcal{E}_0 V^N}{h^{3N} N!} \int_{-\infty}^{\infty} dp^{3N}\, \delta\left(\sum_{i=1}^{3N} p_i^2/2m - \mathcal{E}\right)$$

$p_i = \sqrt{2m}\, y_i$

$$= \frac{\mathcal{E}_0 V^N (2m)^{3N/2}}{h^{3N} N!} \int_{-\infty}^{\infty} dy^{3N}\, \delta\left(\sum_{i=1}^{3N} y_i^2 - \mathcal{E}\right)$$

let $r^2 = \sum_{i=1}^{3N} y_i^2$

a) eq 3: $\int dy^{3N} f(y) = S_{3N-1} \int_0^{\infty} dr\, f(r)\, r^{3N-1}$

eq 4 $\quad = \frac{2\pi^{(3N-1)/2}}{\Gamma(\frac{3N}{2})} \int_0^{\infty} dr\, f(r)\, r^{3N-1}$

so

$$\Omega(N,V,\mathcal{E}) = \frac{\mathcal{E}_0 V^N (2m)^{3N/2}}{h^{3N} N!} \cdot \frac{2\pi^{\left(\frac{3N}{2}\right)}}{\Gamma\left(\frac{3N}{2}\right)} \int_0^{\infty} dr\, \delta(r^2 - \mathcal{E})\, r^{3N-1}$$

b) $$\int_0^\infty \delta(r^2-\varepsilon)\, r^{3N-1} = \frac{1}{2\sqrt{\varepsilon}} \cdot \left(\int_0^\infty \underbrace{\delta(r-\sqrt{\varepsilon})}_{=0,\ -\sqrt{\varepsilon}\text{ outside range}} r^{3N-1} + \int_0^\infty \delta(r+\sqrt{\varepsilon})\, r^{3N-1} \right)$$

so $$\int_0^\infty dr\, \delta(r^2-\varepsilon)\, r^{3N-1} = \frac{1}{2\sqrt{\varepsilon}} \int_0^\infty \delta(r+\sqrt{\varepsilon})\, r^{3N-1}\, dr$$

$$= \frac{1}{2\sqrt{\varepsilon}} (\sqrt{\varepsilon})^{3N-1} = \frac{1}{2\varepsilon} \cdot \varepsilon^{(3N)/2}$$

therefore

$$\Omega(N,V,\varepsilon) = \frac{\varepsilon_0 V^N (2m)^{3N/2}}{h^{3N} N!} \cdot \frac{\cancel{2}\,\pi^{\left(\frac{3N}{2}\right)}}{\Gamma\left(\frac{3N}{2}\right)} \cdot \frac{1}{\cancel{2}\varepsilon} \cdot \varepsilon^{(3N/2)}$$

$$= \left(\frac{\varepsilon_0}{\varepsilon}\right) \frac{V^N (2m)^{3N/2}}{h^{3N} N!} \frac{\pi^{\left(\frac{3N}{2}\right)}}{\Gamma\left(\frac{3N}{2}\right)} \varepsilon^{3N/2}$$

$$\Omega(N,V,\varepsilon) = \frac{\varepsilon_0}{\varepsilon} \frac{1}{N!\,\Gamma\left(\frac{3N}{2}\right)} \left[V \left(\frac{2\pi m \varepsilon}{h^2} \right)^{3/2} \right]^N$$

c)

$$\Omega(N,V,\mathcal{E}) = \frac{\mathcal{E}_0}{\mathcal{E}} \frac{1}{N!\,\Gamma\left(\frac{3N}{2}\right)} \left[V\left(\frac{2\pi m \mathcal{E}}{h^2}\right)^{3/2}\right]^N$$

$$\Gamma\left(\frac{3N}{2}\right) = \left(\frac{3N}{2} - 1\right)! \approx \left(\frac{3N}{2}\right)! \approx \left(\frac{3N}{2}\right)^{\frac{3N}{2}} e^{-3N/2}$$

And $\mathcal{E}^{3N/2 - 1} \approx \mathcal{E}^{3N/2}$

$$\text{so } \Omega(N,V,\mathcal{E}) \approx \frac{\mathcal{E}_0}{N!} \cdot \frac{1}{\left(\frac{3N}{2}\right)^{\frac{3N}{2}} e^{-\frac{3N}{2}}} \cdot \left[V\left(\frac{2\pi m \mathcal{E}}{h^2}\right)^{3/2}\right]^N$$

$$= \frac{\mathcal{E}_0}{N!} \cdot \left[V \cdot \left(\frac{4\pi m \mathcal{E} e}{3h^2 N}\right)^{3/2}\right]^N$$

d) $\mathcal{E} = \frac{3}{2} N k_B T$

1) so $\Omega(N,V,\mathcal{E}) \approx \frac{\mathcal{E}_0}{N!} \cdot \left[V\left(\frac{2\pi m k_B T}{h^2}\right)^{3/2}\right]^N \cdot e^{\frac{3N}{2}}$

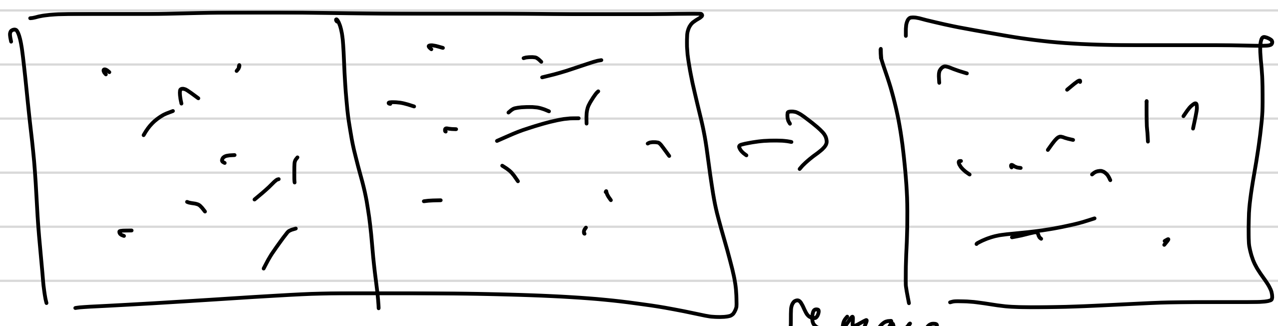
$$S = k_B \ln \Omega \approx N k_B \ln\left[V\left(\frac{2\pi m k_B T}{h^2}\right)\right] + \frac{3N}{2} k_B - k_B \ln N! \checkmark$$

ii) $-k_B \ln N! \approx -k_B [N \ln N - N]$

$$S \approx N k_B \ln\left[\frac{V}{N}\left(\frac{2\pi m k_B T}{h^2}\right)\right] + \frac{5N}{2} k_B \checkmark$$

4) Gibbs paradox, entropy of mixing

what if we didn't have $1/N!$



indistinguishable

$$S_{cl}^{1} \sim N_1 k \log V_1 + 3/2 N_1 k \quad + \text{const} \cdot N_1$$

$$S_{cl}^{2} \quad N_2 k \log V_2 + 3/2 N_2 k \quad + \text{const} \cdot N_2$$

$$S_{cl}^{f} = (N_1+N_2) k \log (V_1+V_2) + \tfrac{3}{2}(N_1+N_2) k \quad + \text{const}(N_1+N_2)$$

$$\Delta S_{cl} = N_1 k \log (V/V_1) + N_2 k \log (V/V_2) > 0 \quad \text{since}$$

$$V > V_1 \ \& \ V > V_2$$

However, w/ correction

$$\Delta S = (N_1+N_2) k \log \left(\frac{V_1+V_2}{N_1+N_2}\right) + \tfrac{5}{2}(N_1+N_2) k$$

$$- (N_1 k \log V_1/N_1 + N_2 k \log V_2/N_2 + 5/2 (N_1+N_2) k)$$

$$= N_1 k \log \frac{V}{N} \frac{N_1}{V_1} + N_2 k \log V/N \, N_2/V_2 \approx 0$$

because we start at equilibrium, where

$$N/V = N_1/V_1 = N_2/V_2 = P/k_B T$$