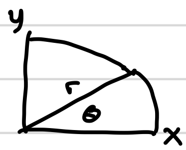
Homework #1 — Solutions

2.1) $I = \int_{-\infty}^{\infty} dx\, e^{-ax^2} = 2\int_0^{\infty} e^{-ax^2}\, dx$ b/c even function 

$$I^2 = 4\int_0^{\infty} dx \int_0^{\infty} dy\, e^{-ax^2} e^{-ay^2} = 4\int dx\,dy\, e^{-a(x^2+y^2)}$$



can switch to polar coordinates,

$$x = r\cos\theta \qquad \Rightarrow \qquad x^2+y^2 = r^2$$
$$y = r\sin\theta \qquad\qquad dx\,dy = r\,dr\,d\theta$$

Jacobian factor

$$\begin{vmatrix} \frac{dx}{dr} & \frac{dx}{d\theta} \\ \frac{dy}{dr} & \frac{dy}{d\theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r$$

$$I^2 = 4\int_0^{\infty} dr \int_0^{\pi/2} d\theta\; r e^{-ar^2} = 2\pi \cdot \int_0^{\infty} dr\; r e^{-ar^2}$$

$u = ar^2 \quad du = 2ar\,dr$
$r\,dr = du/(2a)$

$$= 2\pi \cdot \int_0^{\infty} du \cdot \frac{1}{2a} e^{-u}$$
$$= \pi/a$$

$$\Rightarrow \boxed{I = \sqrt{\pi/a}}$$

2.2 i) $I(a) = \int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\pi}\, a^{-1/2}$

taking the derivative of both sides

$$\frac{dI(a)}{da} = \int_{-\infty}^{\infty} \frac{d}{da} e^{-ax^2} dx = \frac{d}{da}\left(\sqrt{\pi}\, a^{-1/2}\right)$$

$$\int_{-\infty}^{\infty} -x^2 e^{-ax^2} dx = \sqrt{\pi} \cdot -\tfrac{1}{2} a^{-3/2}$$

$$\Rightarrow \int_{-\infty}^{\infty} x^2 e^{-ax^2} dx = \frac{1}{2}\sqrt{\pi / a^3}$$

ii) what is

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} x^2 \frac{e^{-(x-\mu)^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}}$$

lets not worry about const for now, i.e. solve for

$$\sqrt{2\pi\sigma^2}\, \langle x^2 \rangle = \int_{-\infty}^{\infty} dx\, x^2 e^{-a(x-\mu)^2} \qquad \text{with } a = 1/2\sigma^2$$

substitute $y = x - \mu \quad dy = dx$

$$= \int_{-\infty}^{\infty} dy\, (y+\mu)^2 e^{-ay^2}$$

$$= \int_{-\infty}^{\infty} (y^2 + 2\mu y + \mu^2) e^{-ay^2} dy$$

$$= \underbrace{\int_{-\infty}^{\infty} dy\, y^2 e^{-ay^2}}_{\frac{1}{2}\sqrt{\pi/a^3}} + 2\mu \int_{-\infty}^{\infty} dy\, y e^{-ay^2} dy + \mu^2 \int_{-\infty}^{\infty} e^{-ay^2} dy$$

$$\frac{1}{2}\sqrt{\pi/a^3} \quad + \quad 0 \quad + \quad \mu^2 \sqrt{\pi/a}$$

even function times odd function centered at 0

recall $a = \frac{1}{2\sigma^2}$

$$\langle x^2 \rangle = \frac{1}{\sqrt{2\pi\sigma^2}} \left(\frac{\sqrt{8\sigma^6}}{2} + \mu^2 \sqrt{2\pi\sigma^2} \right)$$

$$\boxed{\langle x^2 \rangle = \sigma^2 + \mu^2}$$

Before getting $\langle x^4 \rangle$ lets get

$$\int x^4 e^{-ax^2} dx = -\frac{d}{da} \int x^2 e^{-ax^2} dx = -\frac{d}{da}\left(\tfrac{1}{2}\sqrt{\pi}\, a^{-3/2} \right)$$

$$= \tfrac{3}{4}\sqrt{\pi}\, a^{-5/2}$$

$$\sqrt{2\pi\sigma^2}\,\langle x^4\rangle = \int_{-\infty}^{\infty} dx\, x^4 e^{-(x-\mu)^2/2\sigma^2}$$

again sub $y = x-\mu$, $a = \frac{1}{2\sigma^2}$

$2\sigma^2 = 1/a$

$$= \int_{-\infty}^{\infty} dy\, (y+\mu)^4 e^{-ay^2}$$

$$= \int dy \left(y^4 + 4y^3\mu + 6y^2\mu^2 + 4y\mu^3 + \mu^4 \right) e^{-ay^2}$$

odd integrals vanish, for y & y^3

$$= \int dy\, y^4 e^{-ay^2} + 6\mu^2 \int dy\, y^2 e^{-ay^2} + \mu^4 \int dy\, e^{-ay^2}$$

$$\sqrt{\pi}\,a^{-1/2}\langle x^4\rangle = 3/4\,\sqrt{\pi}\,a^{-5/2} + 6\mu^2 \cdot \frac{1}{2}\sqrt{\pi}\,a^{-3/2} + \mu^4\sqrt{\pi}\,a^{-1/2}$$

$$\Rightarrow \langle x^4\rangle = \frac{3}{4}a^{-2} + 3\mu^2 a^{-1} + \mu^4$$

$$= 3/4 \cdot (2\sigma^2)^2 + 3\mu^2(2\sigma^2) + \mu^4$$

$$= \underline{3\sigma^4 + 6\mu^2\sigma^2 + \mu^4}$$

2.2 ii, when $\mu = 0$

$$\langle x^2 \rangle = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} x^2 e^{-x^2/2\sigma^2} dx = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot \frac{1}{2}\sqrt{\frac{\pi}{a^3}}$$

$a = 1/2\sigma^2$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \cdot \sqrt{2\pi\sigma^6}$$

$$= \sigma^2$$

$$\langle x^4 \rangle = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} x^4 e^{-x^2/2\sigma^2} dx = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot \frac{3}{4}\sqrt{\pi}\, a^{-5/2}$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \frac{3}{4}\sqrt{\pi} \left(2\sigma^2\right)^{5/2}$$

$$= \frac{3}{4}\left(2\sigma^2\right)^{4/2}$$

$$= 3\sigma^4$$

3) a) $\langle e^{x} \rangle = \langle e^{x+\mu-\mu} \rangle$

$= e^{\mu} \langle e^{x-\mu} \rangle = e^{\langle x \rangle} \langle e^{x-\mu} \rangle$

b) $f(x) \equiv e^{x} - x - 1$

$\left.\frac{df}{dx}\right|_{x=0} = e^{x} - 1 \Big|_{x=0} = 1 - 1 = 0$

slope $= 0$, max or min

$\left.\frac{d^2 f}{dx^2}\right|_{x=0} = \frac{d}{dx}\left(\frac{df}{dx}\right)\Big|_{x=0} = \frac{d}{dx}\left(e^{x} - 1\right)\Big|_{x=0}$

$= e^{x}\Big|_{x=0} = e^{0} = 1$

second deriv > 0 means this is a minimum

[also $\frac{df}{dx} = e^{x} - 1 = 0$ only has one solution @ $x = 0$, so, global min.]

$f(x) = e^x - x - 1$ is min at $x = 0$

$f(0) = e^0 - 0 - 1 = 0$

so $f(x) \geq 0$

$\Rightarrow e^x - x - 1 \geq 0$

$\Rightarrow e^x \geq x + 1$

step (a)

(c) $\langle e^x \rangle = e^{\langle x \rangle} \langle e^{x - \mu} \rangle$

step (b)

$\geq e^{\langle x \rangle} \langle x - \mu + 1 \rangle$

$\langle x - \mu + 1 \rangle = (\mu - \mu + 1) = 1$

$\Rightarrow \langle e^x \rangle \geq e^{\langle x \rangle}$ ✓