1a) for $d=2$,

$$V_2 = \pi, \quad S_1 = 2\pi$$

$$S_1/V_2 = 2 = d \checkmark$$

$$V_3 = 4/3\pi \quad S_2 = 4\pi$$

$$S_2/V_3 = 3 = d \checkmark$$

b) i) $I = \int_{-\infty}^{\infty} dx\, e^{-x^2}$

$$I^d = \prod_{i=1}^{d} \int_{-\infty}^{\infty} dx_i\, e^{-x_i^2}$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots dx_1 dx_2 \ldots e^{-\overbrace{\sum_{i=1}^{d} x_i^2}^{r^2}}$$

$$I^d = \int_{-\infty}^{\infty} dx^d e^{-r^2} = S_{d-1} \overline{\int_0^{\infty} r^{d-1} \underbrace{e^{-r^2}}_{f(r)} dr}$$

ii) $\int_0^{\infty} r^{d-1} e^{-r^2} dr = \int_0^{\infty} dy\, r^{d-2} e^{-y} = \int_0^{\infty} dy\, y^{\frac{d-2}{2}} e^{-y}/2$

$y = r^2 \quad dy = 2r\,dr \Rightarrow dr = \frac{dy}{2r}$

$$= \frac{1}{2}\int_0^{\infty} dy\, e^{-y} y^{\frac{d}{2}-1} = \frac{1}{2}\Gamma(d/2)$$

iii) $I = \sqrt{\pi}$ so $I^d = \pi^{d/2}$

$$\Rightarrow \pi^{d/2} = \frac{1}{2}\Gamma(d/2) \cdot S_{d-1}$$

$$\Rightarrow S_{d-1} = \frac{2\pi^{d/2}}{\Gamma(d/2)}$$

iv) $V_d = S_{d-1}/d$

$$\Rightarrow V_d = \frac{2\pi^{d/2}}{d\,\Gamma(d/2)} = \frac{\pi^{d/2}}{(d/2)\Gamma(d/2)} = \underline{\frac{\pi^{d/2}}{\Gamma(d/2+1)}}$$

c) i) $\Gamma(t) = \int_0^\infty dx\, e^{-x} x^{t-1}$

$$\Gamma(1/2) = \int_0^\infty dx\, e^{-x} x^{-1/2} = \int_0^\infty 2r\,dr\, e^{-r^2} r^{-1} =$$

$x = r^2$, $dx = 2r\,dr$

$$= 2\int_0^\infty dr\, e^{-r^2} = \int_{-\infty}^{\infty} dr\, e^{-r^2} = \underline{\sqrt{\pi}}$$

ii) $V_3 = \pi^{3/2}/\Gamma(3/2+1)$

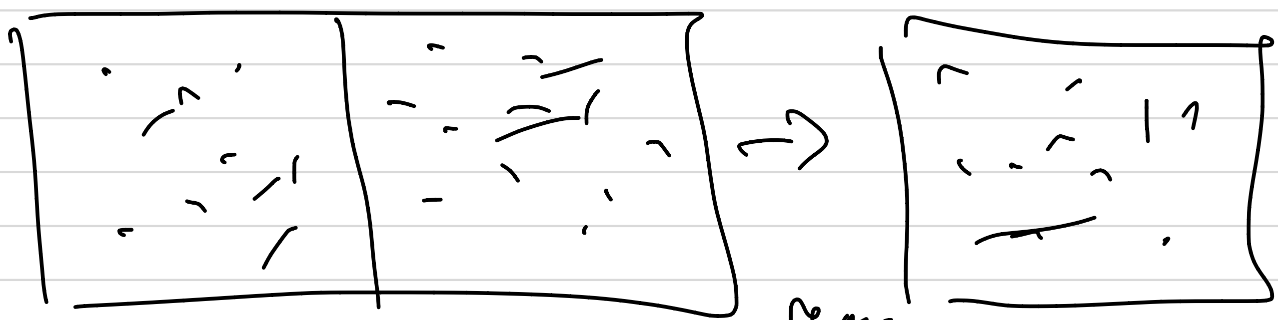
$$\Rightarrow \Gamma(3/2+1) = 3/2\, \Gamma(1/2+1) = 3/2 \cdot \tfrac{1}{2}\, \Gamma(1/2) = \tfrac{3}{4}\sqrt{\pi}$$

$$\Rightarrow V_3 = 4/3\, \pi \quad \checkmark$$

remember
$\Gamma(N+1) = N!$

2) Gibbs paradox, entropy of mixing

what if we didn't have $1/N!$



indistinguishable

$S_{cl}^1 \sim N_1 k \log V_1 + 3/2\, N_1 k$

$S_{cl}^2 \quad N_2 k \log V_2 + 3/2\, N_2 k$

$S_{cl}^f = (N_1+N_2) k \log(V_1+V_2) + \frac{3}{2}(N_1+N_2)k$

$\Delta S_{cl} = N_1 k \log(V/V_1) + N_2 k \log(V/V_2) > 0$ since

$V > V_1$ & $V > V_2$

However, w/ correction

$\Delta S = (N_1+N_2) k \log\left(\frac{V_1+V_2}{N_1+N_2}\right) + \frac{5}{2}(N_1+N_2)k$

$- (N_1 k \log V_1/N_1 + N_2 k \log V_2/N_2 + 5/2\,(N_1+N_2)k)$

$= N_1 k \log \frac{V}{N}\frac{N_1}{V_1} + N_2 k \log V/N\, N_2/V_2 \approx 0$

because we start at equilibrium, where

$N/V = N_1/V_1 = N_2/V_2 = P/k_B T$

3)

a) $$q = \sum_{n=0}^{1} e^{-\beta \varepsilon_n} = e^{-\beta \varepsilon_0} + e^{-\beta \varepsilon_1} = e^{0} + e^{-\beta \varepsilon} = 1 + e^{-\beta \varepsilon}$$

b) A state is the configuration of all spins: $\vec{X} = \{n_1, n_2, \ldots, n_N\}$, where $n_i = 0$ or 1. There are 2^N states. $Q = \sum_{i=1}^{2^n} e^{-\beta E(X_i)}$

we can rewrite as

$$Q = \sum_{n_1=0}^{1} \sum_{n_2=0}^{1} \sum_{n_3=0}^{1} \cdots \sum_{n_N=0}^{1} e^{-\beta \sum_{n=0}^{N} \varepsilon_n}$$

$$= \sum_{n_1=0}^{1} \sum_{n_2=0}^{1} \cdots \sum_{n_N=0}^{1} e^{-\beta \varepsilon_{n_1}} e^{-\beta \varepsilon_{n_2}} \cdots e^{-\beta \varepsilon_{n_N}}$$

$$= \underbrace{\left(\sum_{n_1=0}^{1} e^{-\beta \varepsilon_{n_1}}\right)}_{q} \underbrace{\left(\sum_{n_2=0}^{1} e^{-\beta \varepsilon_{n_2}}\right)}_{q} \cdots \underbrace{\left(\sum_{n_N=0}^{1} e^{-\beta \varepsilon_{n_N}}\right)}_{q} = q^N$$

c) $$\langle \textstyle\sum n_i \rangle = \sum_{i=1}^{N} \langle n_i \rangle = N \langle n \rangle = N \left(\sum_{n=0}^{1} n e^{-\beta \varepsilon_n} / q \right)$$

independent systems

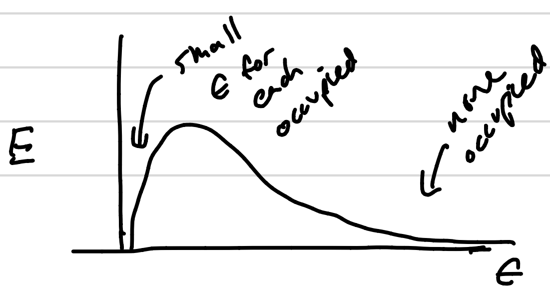
$$= N\left(0 e^{-0} + 1 e^{-\beta \varepsilon} / q \right) = N e^{-\beta \varepsilon} / (1 + e^{-\beta \varepsilon})$$

$$= \frac{N}{(1 + e^{+\beta \varepsilon})}$$

d) $$A = -kT \ln Q_N = -kT N \ln q$$

$$= +kTN \ln\left(1 + e^{-\varepsilon / k_B T}\right)$$

$$E = -\partial/\partial\beta \log Q = -\frac{\partial}{\partial \beta} N \log\left(1 + e^{-\beta \varepsilon}\right)$$



$$= -N \cdot \frac{1}{1 + e^{-\beta \varepsilon}} \cdot -\varepsilon e^{-\beta \varepsilon}$$

$$= N \varepsilon \frac{e^{-\beta \varepsilon}}{1 + e^{-\beta \varepsilon}} = N \varepsilon \cdot \frac{1}{e^{+\beta \varepsilon} + 1}$$

$$S = k \log Q + kT \frac{\partial}{\partial T} \log Q$$

$$= Nk \log(1+e^{-\beta\epsilon}) + kT \cdot N \frac{\partial}{\partial T} \log(1+e^{-\beta\epsilon})$$

$$+ NkT \cdot \frac{1}{1+e^{-\beta\epsilon}} \frac{\partial}{\partial T} e^{-\epsilon/kT}$$

$$+ NkT \cdot \frac{1}{(\cancel{+}e^{-\beta\epsilon}} \cdot -\frac{\epsilon}{k} \cdot -\frac{1}{T^2} e^{-\epsilon/kT}$$

$$+ N/T \cdot \frac{\epsilon}{1+e^{+\beta\epsilon}}$$

$$= Nk \log(1+e^{-\beta\epsilon}) + Nk\beta\epsilon / 1+e^{1}$$

$$C = \frac{-1}{kT^2}\left(\frac{\partial E}{\partial \beta}\right) = -N \frac{1}{k_B T^2} \epsilon \frac{\partial}{\partial \beta} (1+e^{+\beta\epsilon})^{-1}$$

$$= N \frac{1}{k_B T^2} \epsilon \cdot \frac{1}{(1+e^{+\beta\epsilon})^2} \cdot + \epsilon e^{+\beta\epsilon}$$

$$= N \frac{k_B}{(k_B T)^2} \cdot \epsilon^2 \cdot \frac{e^{\beta\epsilon}}{(1+e^{\beta\epsilon})^2}$$

$$= Nk_B \beta^2 \epsilon^2 \; e^{\beta\epsilon} / (1+e^{\beta\epsilon})^2$$