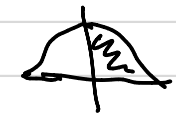
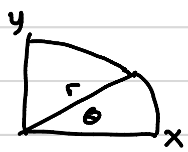
Homework #1 — Solutions

2.1) $I = \int_{-\infty}^{\infty} dx\, e^{-ax^2} = 2\int_0^{\infty} e^{-ax^2} dx$ b/c even function 

$$I^2 = 4\int_0^{\infty} dx \int_0^{\infty} dy\, e^{-ax^2} e^{-ay^2} = 4\int dx\,dy\, e^{-a(x^2+y^2)}$$



can switch to polar coordinates,

$$x = r\cos\theta \qquad y = r\sin\theta \quad \Rightarrow \quad x^2+y^2 = r^2, \qquad dx\,dy = r\,dr\,d\theta$$

Jacobian factor

$$\begin{vmatrix} \frac{dx}{dr} & \frac{dx}{d\theta} \\ \frac{dy}{dr} & \frac{dy}{d\theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r$$

$$I^2 = 4\int_0^{\infty} dr \int_0^{\pi/2} d\theta\, r e^{-ar^2} = 2\pi \cdot \int_0^{\infty} dr\, r e^{-ar^2}$$

$u = ar^2 \quad du = 2ar\,dr \quad r\,dr = du/(2a)$

$$= 2\pi \cdot \int_0^{\infty} du \cdot \frac{1}{2a} e^{-u}$$

$$= \pi/a$$

$$\Rightarrow \boxed{I = \sqrt{\pi/a}}$$

2.2 i) $I(a) = \int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\pi}\, a^{-1/2}$

taking the derivative of both sides

$$\frac{dI(a)}{da} = \int_{-\infty}^{\infty} \frac{d}{da} e^{-ax^2} dx = \frac{d}{da}\left(\sqrt{\pi}\, a^{-1/2}\right)$$

$$\int_{-\infty}^{\infty} -x^2 e^{-ax^2} dx = \sqrt{\pi} \cdot -\tfrac{1}{2} a^{-3/2}$$

$$\Rightarrow \int_{-\infty}^{\infty} x^2 e^{-ax^2} dx = \frac{1}{2}\sqrt{\pi/a^3}$$

ii) what is $\langle x^2 \rangle = \int_{-\infty}^{\infty} x^2 \frac{e^{-(x-\mu)^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}}$

lets not worry about const for now, i.e. solve for

$$\sqrt{2\pi\sigma^2}\,\langle x^2 \rangle = \int_{-\infty}^{\infty} dx\, x^2 e^{-a(x-\mu)^2} \qquad \text{with } a = 1/2\sigma^2$$

substitute $y = x - \mu \quad dy = dx$

$$= \int_{-\infty}^{\infty} dy\, (y+\mu)^2 e^{-ay^2}$$

$$= \int_{-\infty}^{\infty} (y^2 + 2\mu y + \mu^2) e^{-ay^2}\, dy$$

$$= \underbrace{\int_{-\infty}^{\infty} dy\, y^2 e^{-ay^2}}_{\frac{1}{2}\sqrt{\pi/a^3}} + 2\mu \int_{-\infty}^{\infty} dy\, y e^{-ay^2} dy + \mu^2 \int_{-\infty}^{\infty} e^{-ay^2} dy$$

$$\frac{1}{2}\sqrt{\pi/a^3} \quad + \quad 0 \quad + \quad \mu^2 \sqrt{\pi/a}$$

← even function times odd function centered at 0

recall $a = \frac{1}{2\sigma^2}$

$$\langle x^2 \rangle = \frac{1}{\sqrt{2\pi\sigma^2}} \left(\frac{\sqrt{8\sigma^6}}{2} + \mu^2 \sqrt{2\pi\sigma^2} \right)$$

$$\boxed{\langle x^2 \rangle = \sigma^2 + \mu^2}$$

Before getting $\langle x^4 \rangle$ lets get

$$\int x^4 e^{-ax^2} dx = -\frac{d}{da} \int x^2 e^{-ax^2} dx = -\frac{d}{da}\left(\tfrac{1}{2}\sqrt{\pi}\, a^{-3/2} \right)$$

$$= \tfrac{3}{4}\sqrt{\pi}\, a^{-5/2}$$

$$\sqrt{2\pi\sigma^2}\,\langle x^4\rangle = \int_{-\infty}^{\infty} dx\, x^4\, e^{-(x-\mu)^2/2\sigma^2}$$

again sub $y = x-\mu$, $a = \frac{1}{2\sigma^2}$

$2\sigma^2 = 1/a$

$$= \int_{-\infty}^{\infty} dy\, (y+\mu)^4\, e^{-ay^2}$$

$$= \int dy\,(y^4 + 4y^3\mu + 6y^2\mu^2 + 4y\mu^3 + \mu^4)\, e^{-ay^2}$$

odd integrals vanish, for y & y^3

$$= \int dy\, y^4 e^{-ay^2} + 6\mu^2 \int dy\, y^2 e^{-ay^2} + \mu^4 \int dy\, e^{-ay^2}$$

$$\sqrt{\pi}\, a^{-1/2} \langle x^4\rangle = 3/4\, \sqrt{\pi}\, a^{-5/2} + 6\mu^2 \cdot \frac{1}{2}\sqrt{\pi}\, a^{-3/2} + \mu^4 \sqrt{\pi}\, a^{-1/2}$$

$$\Rightarrow \langle x^4 \rangle = \frac{3}{4} a^{-2} + 3\mu^2 a^{-1} + \mu^4$$

$$= 3/4 \cdot (2\sigma^2)^2 + 3\mu^2 (2\sigma^2) + \mu^4$$

$$= \underline{3\sigma^4 + 6\mu^2\sigma^2 + \mu^4}$$

2.2 ii, when $\mu = 0$

$$\langle x^2 \rangle = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} x^2 e^{-x^2/2\sigma^2} dx = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot \frac{1}{2}\sqrt{\frac{\pi}{a^3}}$$

$a = 1/2\sigma^2$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \cdot \sqrt{2\pi\sigma^6}$$

$$= \sigma^2$$

$$\langle x^4 \rangle = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} x^4 e^{-x^2/2\sigma^2} dx = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot \frac{3}{4}\sqrt{\pi}\, a^{-5/2}$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \frac{3}{4}\sqrt{\pi}\left(2\sigma^2\right)^{5/2}$$

$$= \frac{3}{4}\left(2\sigma^2\right)^{4/2}$$

$$= 3\sigma^4$$

3) a) $$\frac{dH^n}{dt} = nH^{n-1}\underbrace{\frac{dH}{dt}}_{0} = 0$$

b) $$\{x,y\} = \sum_{i=1}^{3N} \frac{\partial x}{\partial q_i}\frac{\partial y}{\partial p_i} - \frac{\partial y}{\partial q_i}\frac{\partial x}{\partial p_i}$$

$$\{A(a+b),c\} = \sum_{i=1}^{3N} \frac{\partial(A(a+b))}{\partial q_i}\frac{\partial c}{\partial p_i} - \frac{\partial c}{\partial q_i}\frac{\partial(A(a+b))}{\partial p_i}$$

$$= A\sum_{i=1}^{3N}\left[\frac{\partial a}{\partial q_i}\frac{\partial c}{\partial p_i} - \frac{\partial c}{\partial q_i}\frac{\partial a}{\partial p_i}\right] + \left[\frac{\partial b}{\partial q_i}\frac{\partial c}{\partial p_i} - \frac{\partial c}{\partial q_i}\frac{\partial b}{\partial p_i}\right]$$

$$= A(\{a,c\} + \{b,c\})$$

$$= \boxed{A\{a,c\} + A\{b,c\}}$$

c) The previous result generalizes to

$$\{\sum f_i, c\} = \sum \{f_i, c\}$$

$$\frac{dA}{dt} = \{A, H\} \quad \text{for any } A$$

so e.g. $$\frac{d(H^n)}{dt} = \{H^n, H\}$$

but part (a) says $\frac{dH^n}{dt} = 0$

$$\Rightarrow \{H^n, H\} = 0$$

if $$F(H) = \sum_{n=-\infty}^{\infty} c_n H^n$$

$$\frac{dF}{dt} = \{F, H\} = \{\sum_n c_n H^n, H\}$$

$$= \sum_n c_n \underbrace{\{H^n, H\}}_{0}$$

$$= 0$$