HW7 - Solutions
12/6/2018 - GMH

1) In general for $\frac{dx(t)}{dt} = a\,x(t) + b(t)$

$$\Rightarrow x(t) = e^{at} x(0) + \int_0^t e^{as}\, b(t-s)\, ds$$

in this case

$$\frac{dc}{dt} = -\underbrace{(k_f + k_b)}_{\equiv k} c + \delta F(t), \qquad \langle \delta F(t) \delta F(t') \rangle = 2B\,\delta(t-t')$$

$$\Rightarrow c(t) = e^{-kt} c(0) + \int_0^t e^{-ks}\, \delta F(t-s)\, ds$$

$$\langle c(t)^2 \rangle = e^{-2kt} \langle c(0)^2 \rangle + 2e^{-kt} \int_0^t e^{-ks} \langle c(t)\, \delta F(t-s) \rangle \, dt \quad \xrightarrow{\text{uncorrelated}} 0$$

$$+ \int_0^t \int_0^t e^{-k(s+s')} \langle \delta F(t-s')\, \delta F(t-s) \rangle ds\, ds'$$

$$\langle \delta F(t-s')\, \delta F(t-s) \rangle = 2B\,\delta((t-s')-(t-s))$$

$$= e^{-2kt} \langle c(0)^2 \rangle + \int_0^t e^{-2ks}\, 2B\, ds$$

$$= e^{-2kt} \langle c(0)^2 \rangle + \left(-\frac{e^{-2ks}}{2k} \cdot 2B \right) \Big|_0^t$$

$$= \underbrace{e^{-2kt}}_{\to 0} \langle c(0)^2 \rangle + \frac{B}{k} \left[-\underbrace{e^{-2kt}}_{\to 0} + 1 \right]$$

as $t \to \infty$ $\quad \langle c(t)^2 \rangle \to \langle c^2 \rangle_{eq}$

$$\Rightarrow B = k \langle c^2 \rangle_{eq} = (k_f + k_b) \langle c^2 \rangle_{eq} \ \checkmark$$

2) For ~~Brownian~~ Brownian motion | this case

a)

$$\frac{dx}{dt} = v \qquad\qquad \frac{d\Delta\Theta}{dt} = \Omega$$

$$\frac{dv}{dt} = -\frac{\xi}{m}v + \frac{\delta F(t)}{m} \qquad\qquad \frac{d\Omega}{dt} = -\frac{\xi}{I}\Omega + \frac{1}{I}\delta F(t)$$

$$\langle \delta F(t)\,\delta F(t') \rangle = 2B\delta(t-t') \qquad\qquad \langle \delta F(t)\delta F(t')\rangle = 2\underbrace{\xi k_B T}_{B}\,\delta(t-t')$$

so, comparing to Brownian motion eq for $v(t)$

$$\Omega(t) = \Omega(0)e^{-\xi t/I} + \frac{1}{I}\int_0^t dt'\, e^{-\xi(t-t')/I}\,\delta F(t')$$

$$\langle (\Delta\Theta(t))^2 \rangle = \left\langle \left(\int_0^t \Omega(s)ds\right)^2 \right\rangle$$

$$\frac{d}{dt}\langle(\Delta\Theta(t))^2\rangle = 2\int_0^t \langle \Omega(s)\Omega(t)\,ds\rangle$$

$$= 2\int_0^t \langle \Omega(u)\Omega(0)\rangle ds$$

@ eq time origin doesnt matter

$$\langle \Delta\Theta(t)^2 \rangle = 2\int_0^t \int_0^\tau \langle \Omega(u)\Omega(0)\rangle du\, d\tilde{\tau}$$

$$\langle \Omega(u)\Omega(0)\rangle = \langle \Omega(0)^2\rangle e^{-\xi t/I} + \text{const}\cdot \cancel{\langle \delta F(u)\Omega(0)\rangle}$$

[in class $\langle v^2 \rangle = B/m\xi \Rightarrow B = m\xi\langle v^2\rangle$, here

$$\langle \Omega^2 \rangle = \frac{B}{I\xi} = k_B T/I$$

$$\Rightarrow \langle \Omega(u)\Omega(0)\rangle = k_B T/I\; e^{-\xi t/I}$$

$$\langle \Delta\theta^2 \rangle = 2\int_0^t d\tau \int_0^\tau du \, \frac{k_B T}{I} e^{-\xi u/I}$$

$$\left[\frac{k_B T}{I} \left[\frac{I}{\xi} e^{-\xi u/I} \right]_0^\tau \right]$$

$$= 2\int_0^t d\tau \, \frac{k_B T}{\xi} \left[e^{-\xi\tau/I} + 1 \right]$$

$$= 2k_B T/\xi \left(\left[-\frac{I}{\xi} e^{-\xi \tau/I} \right]_0^t + t \right.$$

$$= \frac{2k_B T}{\xi} \left(t - \frac{I}{\xi} + \frac{I}{\xi} e^{-\xi t/I} \right) \quad \checkmark$$

at large t, $\langle \Delta\theta^2 \rangle \to 2k_B T/\xi \; t$

b)

$$C(t) = \langle e^{i\Delta\theta(t)} \rangle_{eq}$$

$$\langle e^{iax} \rangle = e^{ia\mu - \frac{a^2}{2} \langle (x-\mu)^2 \rangle}$$

for $a = 1$, $x = \Delta\theta(t)$

$\Delta\theta(t)$ random, mean 0

$$C(t) = \langle e^{iax} \rangle_{eq} = e^{-\langle \Delta\theta(t)^2 \rangle_{eq}/2}$$

as $t \to \infty$ $\quad C(t) \to e^{-k_B T/\xi \; t} \quad \checkmark$