1) a) 4 states,

| $s_1$ | $s_2$ | $\varepsilon$ |
|---|---|---|
| 1 | 1 | $-2J-2h$ |
| 1 | -1 | $+2J$ |
| -1 | 1 | $+2J$ |
| -1 | -1 | $-2J+2h$ |

$$Z = e^{-\beta(-2J-2\,)} + e^{-2\beta J} + e^{-2\beta J} + e^{-\beta(-2J+2h)}$$

$$= e^{2\beta(J+h)} + e^{2\beta(J-h)} + 2e^{-2\beta J}$$

b) $$P = \begin{pmatrix} e^{\beta(J+h)} & e^{-\beta J} \\ e^{-\beta J} & e^{\beta(J-h)} \end{pmatrix}$$

$$P^2 = \begin{pmatrix} e^{\beta(J+h)} & e^{-\beta J} \\ e^{-\beta J} & e^{\beta(J-h)} \end{pmatrix} \begin{pmatrix} e^{\beta(J+h)} & e^{-\beta J} \\ e^{-\beta J} & e^{+\beta(J-h)} \end{pmatrix}$$

$$= \begin{pmatrix} e^{2\beta(J+h)} + e^{-2\beta J} & e^{\beta h} + e^{-\beta h} \\ e^{\beta h} + e^{-\beta h} & e^{-2\beta J} + e^{2\beta(J-h)} \end{pmatrix}$$

$$\mathrm{Tr}(P^2) = e^{2\beta(J+h)} + e^{2\beta(J-h)} + 2e^{-2\beta J}$$

2a) $Z = \sum_{\{s_i\}} e^{-\beta(\sum_i -Js_is_{i+1} - hs_i)}$

$$\frac{\partial \log Z}{\partial h} = \sum_{\{s_i\}} \frac{1}{Z}\left(\beta \sum_{i=1}^{N} s_i\right) e^{-\beta(\sum_i -Js_is_{i+1} - hs_i)}$$

$$\frac{\partial}{\partial h}\left(\frac{\log Z}{\partial h}\right) = \sum_{\{s_i\}} \frac{Z\left(\beta \sum_{i=1}^{N} s_i\right)^2 e^{-\beta(\sum_i -Js_is_{i+1} - hs_i)} - \left[\left(\beta \sum s_i\right) e^{-\beta(\sum_i -Js_is_{i+1} - hs_i)}\right]^2}{Z^2}$$

$$= \beta^2\left[\left\langle\left(\sum_{i=1}^{N} s_i\right)^2\right\rangle - \left\langle\sum_{i=1}^{N} s_i\right\rangle^2\right]$$

so $\mathrm{Var}\left[\sum_{i=1}^{N} s_i\right] = \frac{1}{\beta^2}\frac{\partial^2 \log Z}{\partial h^2}$

b) $Z \approx \lambda_+^N$, $\lambda_+ = e^{\beta J}\left(\cosh(\beta h) + \sqrt{\sinh^2(\beta h) + e^{-4\beta J}}\right)$

$$\log Z = N \log \lambda_+$$

$$\frac{\partial \log Z}{\partial h} = \frac{N}{\lambda_+} \cdot e^{\beta J}\left[\beta \sinh(\beta h) + \frac{1}{2}\left(\sinh^2(\beta h) + e^{-4\beta J}\right)^{-1/2} \cdot 2\sinh(\beta h)\cdot \beta\cosh(\beta h)\right]$$

$$= N\beta \cdot \frac{\sinh(\beta h)\left(1 + \cosh\beta h\left(\sinh^2(\beta h) + e^{-4\beta J}\right)^{-1/2}\right)}{\cosh(\beta h) + \sqrt{\sinh^2(\beta h) + e^{-4\beta J}}}$$

N

D

$$\frac{\partial^2 \log Z}{\partial h^2} = N\beta \cdot \frac{D' \cdot \left[\beta\cosh\beta h\left(1 + \cosh\beta h\left(\sinh^2(\beta h) + e^{-4\beta J}\right)^{-1/2}\right) + \left[\cosh\beta h \cdot 2\sinh\cosh\beta \cdot \tfrac{-1}{2}(\ )^{-3/2} + \beta\sinh\beta h(\ )^{-1/2}\right]\right] + N'\left(\beta\sinh\beta h + \frac{1}{2}\left(\sinh^2(\beta h) + e^{-4\beta J}\right)^{-1/2} \cdot 2\sinh\beta h\cosh\beta h \cdot \beta\right)}{\left[\cosh(\beta h) + \sqrt{\sinh^2(\beta h) + e^{-4\beta J}}\right]^2}$$

then $h \to 0$, $\sinh(0) = 0$, $\cosh(0) = 1$

$$A = 1 + e^{-2\beta J} \qquad B = 1 + e^{+2\beta J}$$

$$\left.\frac{\partial^2 \log Z}{\partial h^2}\right|_{h=0} = N\beta^2 \left( \frac{A\ B}{A^2} \right)$$ all numerator terms = 0

$$= N\beta^2 \left( \frac{1 + e^{+2\beta J}}{1 + e^{-2\beta J}} \right)$$

$$\chi = N \left( \frac{1 + e^{+2\beta J}}{1 + e^{-2\beta J}} \right)$$

c) $Z = e^{-\beta N J z m^2} \cdot \left[ 2\cosh[\beta(2mJz + h)] \right]^N$

$$\log Z = -\beta N J z m^2 + N \log\left[ 2\cosh\left[ \beta(2mJz + h) \right] \right]$$

$$\frac{\partial \log Z}{\partial h} = N \cdot \frac{1}{2\cosh(\beta(2mJ_z + h))} 2 \sinh(\beta(2mJz + h)) \cdot \beta$$

$$= N\beta \tanh(\beta(2mJ_z + h))$$

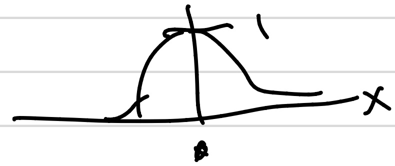
$$\frac{\partial^2 \log Z}{\partial h^2} = N\beta \cdot \mathrm{sech}^2(\beta(2mJ_z + h)) \cdot \beta$$

$$= N\beta^2 \mathrm{sech}^2(\beta(2mJ_z + h))$$

$$\chi = \frac{1}{\beta^2} \frac{\partial^2 \log Z}{\partial h^2} = N \cdot \mathrm{sech}^2(\beta(2mJz + h))$$

To find the max, you would take $\partial \chi / \partial \beta = 0$ and solve for $\beta$

however, plotting $\mathrm{Sech}^2(x)$




we see the max is @ $x = 0$ so

max at $\beta = 0$ or $h = -2mJz$

3) a) $H = \sum_{i=1}^{N} -\epsilon n_i n_{i+1} - \mu n_i$

$$Z = \sum_{n_1=0}^{1} \sum_{n_2=0}^{1} \cdots \sum_{n_N=0}^{1} e^{+\beta \sum_{i=1}^{N}(\epsilon n_i n_{i+1} - \mu n_i)}$$

b) $s_i = 2n_i - 1$  $n_i = 0,\ s_i = -1$

$n_i = 1,\ s_i = +1$

c) $n_i = \frac{s_i + 1}{2}$

$$H = \sum_{i=1}^{N} -\epsilon\left(\frac{s_i+1}{2}\right)\left(\frac{s_{i+1}+1}{2}\right) - \mu\left(\frac{s_i+1}{2}\right)$$

$$= \sum -\frac{\epsilon}{4}\left(s_i s_{i+1} + \underbrace{s_i + s_{i+1}}_{\sum s_i + s_{i+1} = 2\sum s_i} + 1\right) - \mu\left(\frac{s_i}{2} + 1\right)$$

$$= \sum -\frac{\epsilon}{4} s_i s_{i+1} - \frac{\epsilon}{2} s_i - \frac{\mu}{2} s_i - \mu - \frac{\epsilon}{4}$$

$$= \left[\sum -\frac{\epsilon}{4} s_i s_{i+1} - \left(\frac{\epsilon}{2} + \frac{\mu}{2}\right) s_i\right] - N(\mu + \epsilon/4)$$

$$\boxed{J = +\epsilon/4 \qquad h = \frac{\epsilon + \mu}{2}}$$

d) $\langle n | P | n' \rangle = e^{\beta \epsilon n n' + \beta \mu (n + n')/2}$

$\langle 1 | P | 1 \rangle = e^{\beta\epsilon + \beta\mu}$

$\langle 1 | P | 0 \rangle = \langle 0 | P | 1 \rangle = e^{\beta\mu/2}$

$\langle 0 | P | 0 \rangle = 1$

$$P = \begin{pmatrix} e^{\beta(\epsilon+\mu)} & e^{\beta\mu/2} \\ e^{\beta\mu/2} & 1 \end{pmatrix}$$

$$0 = \mathrm{Det}(P - \lambda I) = \mathrm{Det}\begin{bmatrix} e^{\beta(\epsilon+\mu)} - \lambda & e^{\beta\mu/2} \\ e^{\beta\mu/2} & 1-\lambda \end{bmatrix}$$

$$= \left(e^{\beta(\epsilon+\mu)} - \lambda\right)(1-\lambda) - e^{\beta\mu}$$

$$= \underbrace{e^{\beta(\epsilon+\mu)} - e^{\beta\mu}}_{c} - \lambda\underbrace{(1+e^{\beta(\epsilon+\mu)})}_{b} + \underbrace{\lambda^2}_{a}$$

$$\lambda_\pm = \frac{+(1+e^{\beta(\epsilon+\mu)}) \pm \sqrt{(1+e^{\beta(\epsilon+\mu)})^2 - 4\left(e^{(\beta(\epsilon+\mu))} - e^{\beta\mu}\right)}}{2}$$

$$Z = \mathrm{Tr}(P^N) = \lambda_+^N + \lambda_-^N$$