

# HW 4

$$1) \rho(\vec{r}) = \sum_{i=1}^N \delta(\vec{r} - \vec{r}_i)$$

$$\begin{aligned} \rho(\vec{q}) &= \int_V d\vec{r} \rho(\vec{r}) e^{-i\vec{q} \cdot \vec{r}} = \int_V d\vec{r} \sum_{i=1}^N \delta(\vec{r} - \vec{r}_i) e^{-i\vec{q} \cdot \vec{r}} \\ &= \sum_{i=1}^N \int d\vec{r} e^{-i\vec{q} \cdot \vec{r}} \delta(\vec{r} - \vec{r}_i) = \sum_{i=1}^N e^{-i\vec{q} \cdot \vec{r}_i} \end{aligned}$$

$$\begin{aligned} \frac{1}{N} \langle \rho(\vec{q}) \rho(-\vec{q}) \rangle &= \frac{1}{N} \left\langle \left( \sum_{i=1}^N e^{-i\vec{q} \cdot \vec{r}_i} \right) \left( \sum_{j=1}^N e^{+i\vec{q} \cdot \vec{r}_j} \right) \right\rangle \\ &= \frac{1}{N} \left\langle \sum_{j,i} e^{-i\vec{q} \cdot (\vec{r}_j - \vec{r}_i)} \right\rangle = S(\vec{q}) \quad \checkmark \end{aligned}$$

$$2) \text{def } \rho(\vec{q}) = \int d\vec{r} e^{-i\vec{q} \cdot \vec{r}} \left\langle \sum_{i=1}^N \delta(\vec{r} - \vec{r}_i) \right\rangle$$

$$= \left\langle \int d\vec{r} e^{-i\vec{q} \cdot \vec{r}} \sum_{i=1}^N \delta(\vec{r} - \vec{r}_i) \right\rangle$$

$$= \left\langle \sum_{i=1}^N e^{-i\vec{q} \cdot \vec{r}_i} \right\rangle$$

$$\rho(\vec{q} = \vec{0}) = \left\langle \sum_{i=1}^N 1 \right\rangle = \langle N \rangle$$

$$b) S(\vec{q}) \stackrel{!}{=} \frac{1}{\langle N \rangle} \langle \rho(\vec{q}) \rho(-\vec{q}) \rangle$$

$$= \frac{1}{\langle N \rangle} \left\langle \sum_{i,j} e^{-i\vec{q} \cdot (\vec{r}_i - \vec{r}_j)} \right\rangle$$

we can permute  
order is ok b/c  
brackets are integrals  
& sums where  
order can exchange

$$= \frac{1}{\langle N \rangle} \left\langle N + \sum_{i \neq j} e^{-i\vec{q} \cdot \vec{r}_{ij}} \right\rangle$$

$$= \frac{\langle N \rangle}{\langle N \rangle} + \frac{1}{\langle N \rangle} \left\langle \sum_{i \neq j} e^{-i\vec{q} \cdot \vec{r}_{ij}} \right\rangle \quad \downarrow \text{see class}$$

$$= 1 + \frac{1}{\langle N \rangle} \cdot \langle N \cdot (N-1) e^{-i\vec{q} \cdot \vec{r}_{12}} \rangle \quad \checkmark$$

and hence  $S(\vec{q}) = 1 + \frac{1}{\langle N \rangle} \langle N(N-1) \rangle \quad \checkmark = \frac{\langle N^2 \rangle}{\langle N \rangle}$

c)  $\rho k_B T \chi_T = \frac{\text{Var}(N)}{\langle N \rangle} = \frac{\langle N^2 \rangle - \langle N \rangle^2}{\langle N \rangle} = \frac{\langle N^2 \rangle}{\langle N \rangle} - \langle N \rangle = S(\vec{q}=\vec{0}) - \rho(\vec{q}=\vec{0}) \quad \checkmark$

d) now

$$S(\vec{q}) = 1 + \frac{\langle N \rangle}{V} \int_V d\vec{r} g(r) e^{-i\vec{q} \cdot \vec{r}}$$

$$\Rightarrow S(\vec{q}=\vec{0}) = 1 + \frac{\langle N \rangle}{V} \int_V d\vec{r} g(r)$$

$$= 1 + \frac{\langle N \rangle}{V} \int_V d\vec{r} ((g(r)-1) + 1)$$

$$= 1 + \frac{\langle N \rangle}{V} \int_V d\vec{r} (g(r)-1) + \frac{\langle N \rangle}{V} \int_V d\vec{r} 1$$

$$= 1 + \rho \int_V d\vec{r} (g(r)-1) + \langle N \rangle$$

So  $\rho k_B T \chi_T = S(\vec{q}=\vec{0}) - \underbrace{\rho(\vec{q}=\vec{0})}_{\langle N \rangle} = 1 + \rho \int_V d\vec{r} (g(r)-1)$

$$3) a) \quad u(r) = A/r^n = Ar^{-n}$$

low density  $g(r) \approx e^{-\beta u(r)} = e^{-\beta A/r^n}$

$$\beta P = \rho - \frac{2\pi\rho^2}{3} \beta \underbrace{\int_0^\infty dr r^3 \frac{du}{dr} g(r)}_I$$

$$I = \int_0^\infty dr r^3 [-nAr^{-n-1}] e^{-\beta A/r^n}$$

$$= -nA \int_0^\infty dr r^{-n+2} e^{-\beta A r^{-n}}$$

$$\begin{aligned} u(r=0) &= \infty \\ u(r=\infty) &= 0 \end{aligned}$$

$$u = + \beta A r^{-n}$$

$$r = \left( \frac{u}{\beta A} \right)^{-1/n}$$

$$du = + \beta A \cdot (-n r^{-n-1}) dr$$

$$= + \frac{1}{n} \int_\infty^0 du r^3 e^{-u}$$

$$= + \frac{1}{n} \int_\infty^0 du \left[ \frac{u}{\beta A} \right]^{-3/n} e^{-u} = - \frac{1}{\beta} \left[ \frac{1}{\beta A} \right]^{-3/n} \int_0^\infty u^{-3/n} e^{-u}$$

~ flip & get - sign

$$\equiv G$$

$$G = \int_0^{\infty} du u^{-3/n} e^{-u} = \int_0^{\infty} du u^{(-\frac{3}{n}+1)-1} e^{-u}$$

$$= \Gamma(1-3/n)$$

$$\beta P = \rho + \frac{2\pi\rho^2}{3\beta} \left[ \beta A \right]^{3/n} \Gamma(1-3/n)$$

b) If  $n=1$  or  $3$ ,  $\Gamma(1-3/n) = \infty$   
 so the pressure is predicted to be  
 infinite

$$\text{If } n=2, \quad \Gamma(1-3/n) = -2\sqrt{\pi}$$

pressure decreases relative to ideal

unphysical for repulsive potential so prob means  
 approximations not ok here

For  $n > 3$ , pressure increases b/c  $\Gamma(1-3/n) > 0$